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OPTIMAL FEEDBACK CONTROLS FOR PARAMETER IDENTIFICATION.(U)

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OPTIMAL FEEDBACK CONTROLS FOR PARAMETER IDENTIFICATION

Fire Control Branch
Reconnaissance and Weapon Delivery Division

February 1979
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David N Olmstead   Lester McJunkin

FOR THE COMMANDER

R E Deal

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Delivery Division

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This dissertation investigates improving the estimates of unknown constant parameters in the plant and control matrices of a linear discrete system from noisy measurements by the use of a control consisting of a feedback term and open-loop term. The feedback term allows one to move the poles of the system to locations which improve the information in the output about the parameters beyond that attainable with only open-loop control inputs. An energy constraint is placed on the open-loop term of the control and the closed-loop poles are

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required to remain within a predetermined constraint space. Output feedback is used and for the cases where the dimension of the output is less than the dimension of the system states, an additional "consistency" constraint on the closed-loop poles is required. The criterion that has been used is the maximization of the trace or weighted trace of the Fisher information matrix. A gradient projection algorithm has been developed that maximizes this scalar function while maintaining the poles within the constraint space. This procedure results in maximizing the sum of a maximum eigenvalue of a positive semidefinite matrix and a term resulting from the feedback of measurement noise into the process equations. The variable in this maximization procedure is the feedback matrix. The optimal open-loop control sequence is a scaled eigenvector corresponding to the maximum eigenvalue.

The procedure is developed for the multiple parameter and multiple input control cases. Examples are used to demonstrate the enhancement of parameter identification gained by adding feedback control to an open-loop control input.

FOREWORD

The research was carried out while I was assigned with the Fire Control Technology Group, Fire Control Branch, Reconnaissance and Weapon Delivery Division, Air Force Avionics Laboratory. I would like to thank Mr. Gerald Fitzgibbon and Mr. Marvin Spector, and Captains J. Gary Reid and John MacBain for the assistance which they have provided me.

Contents

Contents

List of Figures

List of Symbols

The following list of symbols presents the primary symbols and notation used throughout the research.

Symbol	Meaning	Location
a, b, c	indices	-
$\underline{\bar{a}}$	p-dimensional unknown parameter vector	Eq (3)
$\underline{\hat{a}}$	estimate of $\underline{\bar{a}}$	page 22
$\underline{\hat{a}}'$	average value of $\underline{\hat{a}}$	page 92
$\underline{\bar{a}}_i$	i-th value of $\underline{\bar{a}}$	Eq (175)
$\underline{\bar{a}}_o$	nominal value of $\underline{\bar{a}}$	page 11
$\underline{\bar{a}}_T$	true value of $\underline{\bar{a}}$	Eq (211)
$\underline{c}_i$	i-th column of $\underline{C}$	Eq (157)
$\underline{e}_{max}$	unit length eigenvector of $\underline{W}_N'$ corresponding to $\lambda_m(\underline{F}_y)$	Eq (105)
e_{max_i}	i-th element of $\underline{e}_{max}$	Eq (107)
$\underline{e}_k$	unit length eigenvector corresponding to λ_k	Eq (105)
$\underline{e}_{ij}$	rows of $\underline{\hat{Q}}^{-1}$	page 85
f_{s_i}, $f_{s_{ij}}$	elements of $\underline{F}_s$	Eq (157) Eq (250)
f_{y_i}, $f_{y_{ij}}$	elements of $\underline{F}_y$	Eq (157) Eq (250)
$\underline{g}_{k_i}$	diagonal blocks of $\underline{G}_c(\bar{a})$	page 57
$\underline{h}$	2nxn matrix consisting of zero and identity blocks	Eq (51)
h_{jp}	elements of $\underline{H}$	page 140

i, j, k, l, m, n, p, r	indices	-
k_{ij}	coefficients of consistency equations	Eq (161)
k_s	gradient step size	Eq (175)
k_m	control invariant	Page 59
m_{ik}	elements of $\underline{M}$	Eq (217)
$\bar{\underline{n}}(j)$	vector variable	Eq (29)
$\underline{u}(k)$	m-dimensional control vector	Eq (1)
$\underline{v}(k)$	m-dimensional external control vector	Eq (1)
$\underline{v}_c(k)$	m-dimensional transformed control vector	Eq (133)
$\bar{\underline{w}}(k)$	r-dimensional measurement noise vector	Eq (7)
$\underline{w}_c(k)$	m-dimensional transformed external control vector	page 57
w'_{ij}	element of $\underline{W}'_N$	Eq (83)
$\underline{w}'_{kl_i}$	element of $\underline{W}'_{N_i}$	Eq (227)
$\underline{w}'_{ab_{ij}}$	element of $\underline{w}'_{ab}$	Eq (274)
$\underline{x}(k)$	n-dimensional state vector	Eq (3)
$\hat{\underline{x}}(k)$	estimate of $\underline{x}(k)$	Eq (12)
$\underline{y}(k)$	r-dimensional output vector	Eq (1)
$\underline{z}(k)$	n-dimensional general canonical state vector	Eq (131)
$\underline{A}$	nxn matrix	page 40
$\underline{A}(\bar{\underline{a}})$	nxn plant matrix	Eq (3)
$\underline{A}(\bar{\underline{a}}_o)$	nxn plant matrix with nominal parameter value	Eq (12)

i, j, k, l, m, n, p, r	indicies	-
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$\underline{A}$	nxn matrix	page 40
$\underline{A}(\underline{\bar{a}})$	nxn plant matrix	Eq (3)
$\underline{A}(\underline{\bar{a}}_o)$	nxn plant matrix with nominal parameter value	Eq (12)

Symbol	Meaning	Location
$\underline{A}(\bar{\underline{a}}_T)$	nxn plant matrix with true value of $\bar{\underline{a}}$	Eq (211)
$\underline{A}_A$	2nx2n augmented plant matrix	Eq (47)
$\underline{A}_{A_i}$	2nx2n augmented plant matrix with partials wrt $\bar{a}_i$	Eq (221)
$\underline{A}_F(\bar{\underline{a}})$	nxn feedback plant matrix	Eq (66)
$\underline{B}(\bar{\underline{a}})$	nxm control matrix	Eq (3)
$\underline{B}(\bar{\underline{a}}_o)$	nxm control matrix with nominal parameter value	Eq (12)
$\underline{B}(\bar{\underline{a}}_T)$	nxm control matrix with true value of $\bar{\underline{a}}$	Eq (211)
$\underline{B}_A$	2nxm augmented control matrix	Eq (45)
$\underline{B}_{A_i}$	2nxm augmented control matrix with partials taken wrt $\bar{a}_i$	Eq (219)
$\underline{C}$	rxn output matrix	Eq (4)
$\underline{C}'$	rxr matrix derived from $\underline{C}$	Eq (158)
$\underline{D}_A$	2nxr matrix of feedback times augmented control matrix	Eq (46)
$\underline{D}_{A_i}$	2nxr matrix of feedback times augmented control with partials taken wrt $\bar{a}_i$	Eq (220)
E	energy constraint	Eq (18)
E'	energy constraint with initial state conditions	Page 43
$\underline{E}(k)$	(2n+N)x2n matrix	Eq (74)
$\underline{F}_c(\bar{\underline{a}})$	nxn plant matrix for generalized control canonical form	Eq (131)
$\bar{\underline{F}}_c$	transformed plant matrix	Page 85
$\underline{F}_d$	nxn generalized control canonical feedback plant matrix	Eq (139)

Symbol	Meaning	Location
$\underline{F}'_{d_k}$	companion blocks of $\underline{F}_d$	Page 58
$\underline{F}_{k_i}$	companion blocks of $\underline{F}_c(\bar{\underline{a}})$	Page 57
$\underline{F}_e$	mxn state feedback matrix	Page 55
$\underline{F}_y$	mxr output feedback matrix	Eq (1)
$\underline{F}_{yc}$	current value of $\underline{F}_y$	Eq (101)
$\underline{F}_{ynew}$	updated $\underline{F}_y$	Page 104
$\underline{F}_{yo}$	initial value of $\underline{F}_y$	Page 33
$\underline{F}_{yopt}$	optimal value of $\underline{F}_y$	Page 43
$\underline{G}_c(\bar{\underline{a}})$	nxm generalized control canonical form of control matrix	Eq (131)
$\bar{\underline{G}}_c$	transformed control matrix	Page 85
$\underline{H}$	canonical feedback matrix	Eq (138)
$\underline{I}$	identity matrix	Page 17
$J(\underline{F}_y)$	M term independent of external controls	Page 30
$J'_i(\underline{F}_y)$	tr $\underline{M}$ term independent of external controls	Eq (226)
$J''(\underline{F}_y)$	summation of $J'_i(\underline{F}_y)$'s	Eq (230)
K	M term dependent on external controls	Eq (64)
$\bar{\underline{K}}(k)$	matrix in Kalman filter Eqs	Eq (12)
$\underline{L}$	mxn generalized control canonical feedback matrix	Page 57
$L(\bar{\underline{a}})$	negative term of likelihood function	Eq (173)
$L(\hat{\underline{a}})$	likelihood function evaluated at $\underline{a} = \hat{\underline{a}}$.	Eq (174)
$L'(\bar{\underline{a}})$	likelihood function	Eq (172)
$\underline{M}$	Fisher information matrix	Eq (5)

Symbol	Meaning	Location
$\underline{M}_k$	information matrix for control $\underline{V}_k$	Eq (234)
$\underline{M}_m(\underline{F}_y)$	maximum value of $\underline{M}$ for given $\underline{F}_y$	Eq (87)
$\underline{M}_o$	information matrix for control $\underline{V}_o$	Page 107
$\overline{M}_o$	initial value of M	Eq (121)
M_{opt}	optimal information scalar	Page 42
N	final time step	Page 13
$\underline{P}(k)$	nxn state covariance matrix	Page 16
$\underline{P}_A(k)$	2nx2n augmented state covariance matrix	Eq (50)
$\underline{P}_{A_{ij}}(k)$	2nx2n augmented covariance matrix with partials wrt $\bar{a}_i$ and $\bar{a}_j$	Eq (223)
$\underline{Q}$	plant noise covariance matrix	Page 16
$\overline{\underline{Q}}$	controllability matrix	Eq (194)
$\hat{\underline{Q}}$	controllability matrix	Page 84
$\underline{R}$	measurement noise covariance matrix	Eq (8)
$\overline{R}$	radius of constraint space	Eq (122)
$\underline{S}$	weighting matrix	Page 102
$\underline{T}$	state transformation matrix	Page 57
$\underline{V}$	external control sequence	Page 22
$\underline{V}_k$	iterative value for $\underline{V}$	Page 108
$\underline{V}_m$	$\underline{V}$ that maximizes tr $\underline{M}_o^{-1}\underline{M}$	Page 107
$\underline{V}_N$	state initial conditions plus external control sequence	Eq (226)
$\underline{V}_{new}$	updated $\underline{V}$	Page 104
$\underline{V}_o'$	value of $\underline{V}$ before $\lvert\underline{M}^{-1}\rvert$ increases	Page 107

Symbol	Meaning	Location
$\underline{V}_{opt}$	optimal value of $\underline{V}$	Page 43
$\underline{V}_X(k)$	initial state vector plus control sequence to time k-1	Eq (73)
$\underline{V}_{X_A}(k)$	initial augmented state vector plus controls to time k-1	Eq (71)
$\underline{W}(k)$	(n+k)x(n+k) matrix dependent on $\underline{F}_y$	Eq (77)
$\underline{W}_{j-1}$	sequence of measurement noise	Page 24
$\underline{W}_N$	$\underline{W}(k)$ with k = N	Eq (79)
$\underline{W}_N'$	NxN matrix equal to $\underline{W}_N$ with zero initial states	Page 37
$\underline{W}_{N_i}'$	$\underline{W}_N'$ with partial taken wrt $\bar{a}_i$	Eq (226)
$\underline{W}_N''$	sum of $\underline{W}_{N_i}'$'s	Eq (230)
$\underline{X}$, $\underline{Y}$	general matrices	Page 26
$\underline{X}_A(k)$	2n-dimensional augmented state vector	Eq (44)
$\hat{\underline{X}}_A(k)$	2n-dimensional augmented estimated state vector	Eq (49)
$\underline{X}_{A_i}(k)$	$\underline{X}_A(k)$ with partials wrt $\bar{a}_i$	Eq (218)
$\hat{\underline{X}}_{A_i}(k)$	estimate of $\underline{X}_{A_i}(k)$	Eq (222)
Y_N	sequence of output measurements	Eq (5)
$\underline{Z}$	control transformation matrix	Page 57
β	scalar variable	Page 108
δ_{kj}	Dirac delta function	Eq (8)
ε	scalar variable	Page 108
λ	eigenvalue variable	Eq (144)
$\underline{\lambda}_F$	n-dimensional eigenvalue vector of $\underline{A}_F(\bar{\underline{a}})$	Page 54

Symbol	Meaning	Location
λ_{F_i}	i-th eigenvalue of $\underline{\lambda}_F$	Eq (142)
$\underline{\lambda}_{Fc}$	current value of $\underline{\lambda}_F$	Eq (163)
$\underline{\lambda}_{Fo}$	initial value of $\underline{\lambda}_F$	Page 70
λ_k	k-th eigenvalue of $\underline{W}_N'$	Eq (105)
λ_m	maximum eigenvalue of $\underline{A}$	Page 40
$\lambda_{max}(\underline{F}_y)$	maximum eigenvalue of $\underline{W}_N'$	Eq (87)
$\lambda_{max}(\underline{F}_{yopt})$	maximum eigenvalue of $\underline{W}_N'$ with $\underline{F}_y = \underline{F}_{yopt}$	Page 43
λ_{pmax}	maximum eigenvalue of $\underline{W}_N''$	Eq (233)
$\underline{\nu}(k)$	variable vector in Kalman filter equations	Eq (12)
σ	real axis of complex plane	Page 54
$\underline{\phi}(k)$	plant noise vector	Eq (10)
ω	imaginary axis of complex plane	Page 54
$\Delta\underline{F}_y$	incremental step of $\underline{F}_y$	Eq (121)
$\Delta\underline{\lambda}_F$	incremental step of $\underline{\lambda}_F$	Page 71
$\underline{\Lambda}_N$	diagonal matrix of eigenvalues of $\underline{W}_N'$	Eq (104)
$\underline{\Xi}$	positive definite weighting matrix	Eq (168)
$\underline{\Sigma}(k)$	rxr matrix in Kalman filter	Eq (13)
$\underline{\Upsilon}$	NxN orthogonal matrix	Page 46
Φ	constraint space	Page 21
$\underline{\Psi}(\underline{\lambda}_F)$	constraint vector	Eq (168)

I Introduction

The implementation of modern control techniques requires a detailed knowledge of the controlled system. To accomplish this objective, a mathematical model of the system is developed, but this model generally contains parameters which are known only with some limited degree of certainty. It may be necessary for adequate control that these unknown parameters be estimated as accurately as possible.

One method of obtaining the estimates is for the engineer, based on his experience and familiarity with the system, to assign a value to the parameters. This is usually not adequate for most system functions. A better method is to perform experiments on the system and, by observing the inputs, outputs, and other available data, to estimate the parameters. It is well known that the choice of the input signals has a direct bearing on the accuracy of the parameter estimates for most systems. The objective of this research is to improve the estimates of the parameters by modifying the input controls.

A typical controlled system is shown in Figure 1. Uncertainties and disturbances are also generally inputs to the system, but are not addressed directly in this research. The system output measurements are corrupted by noise and the input control signals usually belong to either an open-loop, feedback, or closed-loop policy. The usual definition of these different controls as given by Bar-Shalom and Tse

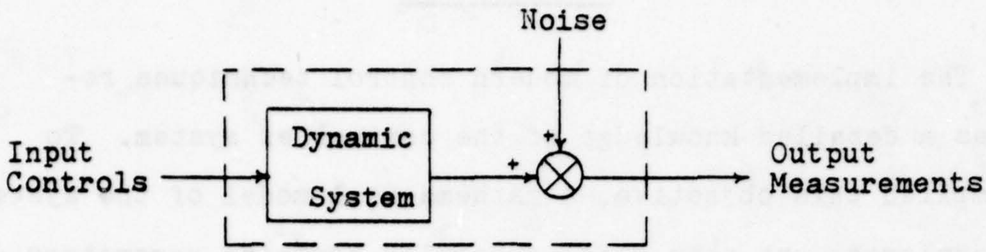


Figure 1. Controlled System Diagram

(Ref 5) will be used in this research. An open-loop control policy is determined from a priori information only. Even if output measurements are taken, this information is not used to calculate the controls. The complete control sequence is determined before the test has begun. A feedback control policy is calculated from a priori information plus any measured data up to and including the time the control is required. It ignores the fact that future measurements will be made, while a closed-loop control policy uses the same information as the feedback plus it exploits the fact that future measurements will be made in its calculation of appropriate controls.

It is shown in the next section that most research has addressed calculating an optimal open-loop control policy that improves the identification of unknown parameters in the system. The advantage of the open-loop controls is that they are usually easier to calculate than feedback and closed-loop controls. An expected disadvantage is that noise and other uncertainties modify the system's response to the controls, so that the control sequence is no longer

optimal.

A method of monitoring the output and adjusting the input control accordingly is provided by the use of a feedback term. The advantages of feedback to control the output with parameter changes is well known, but little work has been done in using feedback to enhance the information in the output about parameter changes. This dissertation develops a method of computing a feedback control policy that improves the identification of unknown parameters in the system beyond that achievable with purely open-loop inputs. The control will be of the discrete-time form

$$\underline{u}(k) = \underline{F}_y\underline{y}(k) + \underline{v}(k) \tag{1}$$

where $\underline{u}(k)$ is an m-dimensional input control at the k-th time instant, $\underline{F}_y$ is a constant mxr feedback matrix, $\underline{y}(k)$ is the r-dimensional output vector at the k-th sample time instant, and $\underline{v}(k)$ is an m-dimensional open-loop control vector at the k-th instant. The terms $\underline{F}_y$ and $\underline{v}(k)$ will be calculated a priori; however, $\underline{u}(k)$ is a feedback control policy because of the output vector, $\underline{y}(k)$, appearing in Eq (1).

Figure 2. shows the modified controlled system. The addition of this feedback term increases the degree of freedom we have to optimize the information available in the output measurements for parameter identification. It is a means of adjusting the system eigenvalues. This may be necessary in order to stabilize an originally unstable

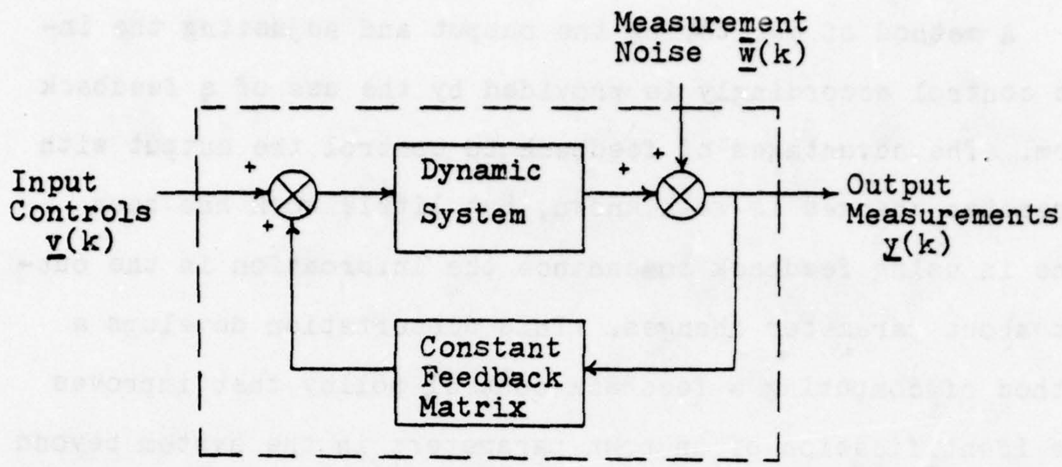


Figure 2. Controlled System with Feedback

system or to make an originally unidentifiable system identifiable. It is shown in this dissertation that the addition of feedback of the output measurements greatly improves the parameter estimates when in the presence of a high measurement noise environment.

To limit the scope of the dissertation, all parameters are assumed constant, and a nominal value for each parameter is known a priori. Only linear time-invariant mathematical models of systems are addressed. These models are the parametric state models, and the initial conditions on the state vector is assumed to be either zero or calculated by the procedure developed in the next chapter. This implies that there is an optimal initial condition for the state vector to improve the parameter estimates. Dynamic driving noise

is not considered in this research. It will be shown in a later section of this chapter that the addition of process noise increases the complexity of the problem.

Background Literature

A considerable amount of research has been performed in the area of optimal inputs for parameter identification. The word "optimal" implies that some criterion must be used to select the desired control. The criterion used in most of the research is either maximization of a scalar function of the Fisher information matrix (Ref 27) or minimization of a scalar function of the parameter error covariance matrix.

The use of the parameter error covariance matrix seems natural since the primary purpose is to obtain the most accurate parameter estimates. Using controls to minimize functions of this error matrix should give us these accurate estimates. Unfortunately, in general, it is mathematically complicated to use this criterion; therefore, the use of the Fisher information matrix became attractive.

The information matrix is a measure of the amount of information about the parameters that is available in the output. Maximizing this matrix in some manner is then a means of increasing the information available about the unknown parameters. However, "maximizing a matrix" usually is accomplished by maximizing a scalar function of the matrix. Some of the function used have been the trace, the weighted trace, and the determinant of the matrix. Some

advantages and disadvantages of using these various functions will be presented when they are first discussed in the following paragraphs.

Most of the previous research has been in the area of developing open-loop inputs for both linear and nonlinear systems with measurement noise. Process noise is usually not considered. The main reasons are that it results in a more difficult math model of the system and the computation required is more involved. When process noise is used, the system considered is usually linear and described by a low dimension model. Levin (Ref 16) considers single-input, single-output linear systems without process noise. His criterion is based on the parameter error covariance matrix. The parameters that are estimated represent the impulse response of the system. His system model is, therefore, linear in the parameters so he uses the least square estimation technique. He shows that the parameter error covariance matrix is made up of input control terms only and that the inverse of this matrix has identical terms on the main diagonal. Levin uses a principal, that he proves in the article, that by making all the off diagonal terms of the inverse matrix zero, he minimizes the diagonal terms of the error matrix. For a stationary random process input with zero mean, he shows that this implies that the input must be white.

Mehra (Ref 20 and 21) considers linear and nonlinear systems with and without process noise. His criterion is the maximizing of the weighted trace of the Fisher informa-

tion matrix. The advantage of using the trace is that it is easier to compute than the determinant, but the main disadvantage is that this criterion could lead to cases in which the system is unidentifiable. This along with a discussion of the possible criteria is given in the section on multiple unknown parameters in Chapter IV. To overcome this problem, the information matrix is multiplied by a weighting matrix in which the weighting factors are properly selected to keep the system identifiable. Mehra (Ref 21) and Gupta and Hall (Ref 11) have shown how to select these weighting factors in such a way that the resulting optimal control minimizes either the determinant or trace of the inverse of the information matrix.

Aoki and Staley (Ref 1) basically consider the same problem. Mehra considers energy constraints on the inputs and similarly, Aoki and Staley also consider energy constraints on the output.

Nahi and Wallis (Ref 23), Wallis (Ref 34), and Napjus (Ref 24) consider nonlinear systems without process noise. Their optimality criterion is the maximizing of the weighted trace of the information matrix. To calculate the information matrix they use augmented sensitivity equations, which are equations relating the sensitivity of the output to parameter changes. Amplitude constraints on the inputs are also incorporated.

Reid (Ref 28) and Goodwin and Payne (Ref 9) use the

minimization of the weighted inverse of the information matrix as their criterion. Using this criterion is more difficult than maximizing the trace of the information matrix, but it does not lead to the problem of unidentifiable parameters.

All of the work referenced above produce open-loop controls that are based on a priori information only. It is found that in cases where the output noise is large or the initial estimates of the unknown parameters are very erroneous, the identification process produces very inaccurate estimates. Gustavsson, Ljung, and Soderstrom state in their survey paper (Ref 12) that feedback terms may have to be included in the input signal, feedback plus open-loop controls, in order to obtain acceptable accuracy. For large disturbances and high noise levels, feedback terms may be needed. Inclusion of feedback will also increase the degree of freedom when choosing the input signals. They further state that it is difficult to determine a priori whether a feedback term is necessary to give better estimates; therefore, the optimal inputs should include feedback terms with the open-loop terms instead of open-loop controls only. Although the addition of feedback potentially enhances the parameter identification significantly, the complexity of solving this generalized problem is substantially greater than that associated with open-loop controls only.

In an attempt to develop a closed-loop control, Keviczky

and Banyasz (Ref 15) and Arimoto and Kimura (Ref 3) approach the identification problem by designing inputs that maximize the incremental increase in information during the next measurement period. The inputs are not true closed-loop controls since they do not take future learning into account. The one step ahead approach is used to simplify the calculations so that an on-line algorithm could be used. Both articles consider an amplitude constraint on the input signal.

Lopez-Toledo (Ref 17) attempts to find the optimal closed-loop input with an energy constraint that will maximize the trace of the expectation of the inverse of the conditional error covariance matrix. He then showed that for his model, linear in the parameters, the "curse of dimensionality" makes the problem intractable. He then develops suboptimal inputs by using an affine law (input is a linear function of the output plus a constant term) and then the open-loop feedback optimal (OLFO) approach. The OLFO method is to compute an optimal open-loop control at each time step, but apply only the first control and then recompute at the next time step. Lopez-Toledo's affine law is the same as the control policy used in this research. The main differences are that this research does not restrict the model to be linear in the parameters, but does restrict the freedom of the closed-loop system eigenvalues. These differences were chosen because most real systems can-

not be put into the linear-in-parameters form and adding feedback moves the eigenvalues which must be constrained to maintain stability.

Upadhyaya and Sorenson (Ref 32 and 33) develop a feedback control by casting the problem as an OLFO control problem. The model used is linear in the parameters, and when the a priori distribution of the parameters is known, the optimal signal is obtained by maximizing the Bhattacharyya coefficients, B_{ij}, where $i \neq j$. The coefficients are defined as

$$B_{ij} = -\ln \int (p(\underline{Z}^k | \underline{\theta}_i) p(\underline{Z}^k | \underline{\theta}_j))^{\frac{1}{2}} dZ^k \tag{2}$$

where $p(\underline{Z}^k | \underline{\theta}_i)$ is the probability distribution of the measurement sequence, $\underline{Z}^k$, given the parameter vector, $\underline{\theta}_i$. $\underline{\theta}$ is assumed to take on only a discrete number of values so B_{ij} is finite dimensional. Maximizing the Bhattacharyya measure maximizes the distance between the pairs of distributions and makes the identification process more accurate. The resultant control is a feedback control.

Goodwin and Payne (Ref 10) develop optimal open-loop and feedback controls by minimizing the negative of the log of the determinant of the information matrix. They show that if there are no unknown parameters that are in both the plant matrix and measurement noise matrix and the total input power, open-loop and feedback controls, is constrained, then the optimal control consists of only an open-loop con-

trol. In this research, however, the energy constraint is only on the open-loop controls and not on the combined controls.

The next section discusses the reasoning behind selecting a control made up of the sum of a linear feedback term and an open-loop term. The section also investigates the different models available and discusses why the linear, time-invariant model with measurement noise only was selected for this research.

Problem Formulation

Consider a linear time-invariant discrete system

$$\underline{x}(k+1) = \underline{A}(\bar{\underline{a}})\underline{x}(k) + \underline{B}(\bar{\underline{a}})\underline{u}(k) \tag{3}$$

$$\underline{y}(k) = \underline{C}\underline{x}(k) \tag{4}$$

where $\underline{x}(k)$ is an n-dimensional state vector at time instant k, $\underline{u}(k)$ is an m-dimensional control vector, and $\underline{y}(k)$ is an r-dimensional output vector. $\underline{A}(\bar{\underline{a}})$ is an nxn time-invariant plant matrix, $\underline{B}(\bar{\underline{a}})$ is an nxm time-invariant control matrix, and $\underline{C}$ is an rxn time-invariant output matrix. The symbol $\bar{\underline{a}}$ represents the p-dimensional unknown parameter vector. As mentioned earlier, it is assumed that the parameters are constant and that a nominal value of the parameter vector $\bar{\underline{a}}$ is known and is represented by $\bar{\underline{a}}_o$. The plant and control matrices are a function of $\bar{\underline{a}}$ while the output matrix is not. This is a reasonable constraint since the uncertainties in sensors are usually less than in a dynamic system or the

actuators. The matrices are not normally linear with respect to the unknown parameters when they represent a physical system. This also occurs when a continuous system is changed to a discrete system and unknown parameters exist in the plant matrix of the continuous system. In the discrete system the unknown parameter will usually appear nonlinearly in the plant and control matrices. The system in this research is considered to be both completely observable and completely controllable. The initial condition of the state vector, $\underline{x}(0)$, will be known. Its value will be either $\underline{0}$ or will be calculated to optimize the estimation technique.

The control is chosen to have the form

$$\underline{u}(k) = \underline{F}_y \underline{y}(k) + \underline{v}(k) \tag{1}$$

that is, the control is a combination of a linear function of the current output and an open-loop control. If $\underline{F}_y = \underline{0}$, then the optimal input control sequence $\underline{u}(k)$ could be calculated from some of the referenced methods for open-loop controls. If $\underline{v}(k) = \underline{0}$, then only the feedback term is left. Using the feedback control $\underline{u}(k) = \underline{F}_y \underline{y}(k)$ to identify the system could give erroneous results. Box and MacGregor (Ref 6) show that identification techniques that are based on this input control, $\underline{u}(k)$, and the output, $\underline{y}(k)$, may identify the pseudoinverse of the feedback matrix, that is, $\underline{F}_y^{+}$. Wellstead (Ref 35) shows that the difficulties associated with identification of systems with feedback controls are most readily solved by the addition of an external signal.

Gustavsson, Ljung, and Soderstrom (Ref 12) state that if the number of independent "extra inputs", $\underline{v}(k)$, is equal to the number of inputs to the system, then the system is always identifiable no matter what the feedback is. It is for these reasons that the input control used in this research is of the form of Eq (1).

The feedback matrix will move the closed-loop system poles, the eigenvalues of $\underline{A}(\bar{\underline{a}}) + \underline{B}(\bar{\underline{a}})\underline{F}_y\underline{C}$, to enhance the identification. Movement of the poles also indicates the possibility of causing the feedback system to be unstable. Constraints must be placed on these poles in order to prevent this from happening. The open-loop control, $\underline{v}(k)$, then not only optimizes the identification, but also eliminates the danger of unidentifiable parameters as mentioned in this section.

The criterion that is used is the maximization of a function of the Fisher information matrix. This criterion is chosen because as mentioned in the previous section, it is mathematically easier to compute and maximizes the information about the parameters in the output. The information matrix is

$$\underline{M} = \underset{Y_N}{E}\left[\frac{\partial \ln p(Y_N|\bar{\underline{a}})}{\partial \bar{\underline{a}}}^T \frac{\partial \ln p(Y_N|\bar{\underline{a}})}{\partial \bar{\underline{a}}}\right] \tag{5}$$

where Y_N is a set of all measurements up to time instant N, $p(b|c)$ is the probability distribution of random variable b

given that c has occured, and $E_{Y_N}[b]$ represents the expectation of b taken over the sample space of observations Y_N. Appendix B outlines the convention that is used when the partial of a scalar is taken with respect to a vector. The conditional likelihood function $\ln p(Y_N|\bar{a})$ must be evaluated, and it is this function that has a bearing on the system model selected in this research.

The function of the information matrix that is maximized in this research is the trace or weighted trace. A weighted trace means that the information matrix is premultiplied by a matrix whose elements are weighting factors and then the trace operation is performed. These weighting factors can be determined a priori or during the iterative process of computing the optimal controls. Experience may indicate that information in certain outputs is more important than in others, so these terms would receive a higher weighting factor than the others. The engineer could assign these values a priori; Chapter IV gives a method for computing the factors during the iteration steps, but as will be shown, this requires much more computation time.

Equations (3) and (4) are deterministic equations that are assumed to model the system. With no process noise and $\underline{x}(0)$ known, the output vector $\underline{y}(k)$ can be measured exactly for any input control sequence $\underline{u}(k-1)$, $\underline{u}(k-2), \cdots$, $\underline{u}(0)$. The only unknown in the equations is $\bar{a}$. If the control sequence is such that the system is identifiable (that is, $\bar{a}$

can be uniquely determined from a given set of experimental observations) then $\underline{\bar{a}}$ can be found by a nonlinear optimization method such as the gradient projection technique. The condition to be imposed on the controls is that they make the system identifiable, that is, the eigenvalues of $\underline{M}$ are nonzero. Maximizing the weighted trace of $\underline{M}$ is not meaningful for this case since there will always be sufficient information in the output to estimate $\underline{\bar{a}}$. For this reason the deterministic case is not addressed further in this research.

If measurement noise is added to the model, then Eqs (3) and (4) become

$$\underline{x}(k+1) = \underline{A}(\underline{\bar{a}})\underline{x}(k) + \underline{B}(\underline{\bar{a}})\underline{u}(k) \tag{6}$$

$$\underline{y}(k) = \underline{Cx}(k) + \underline{\bar{w}}(k) \tag{7}$$

where $\underline{\bar{w}}(k)$ is an r-dimensional discrete time Gaussian white noise vector with

$$E(\underline{\bar{w}}(k)) = \underline{0} \quad \text{and} \quad E(\underline{\bar{w}}(k)\underline{\bar{w}}(j)^T) = \underline{R}\,\delta_{kj} \tag{8}$$

Inserting the control given in Eq (1) into Eqs (6) and (7) yields:

$$\underline{x}(k+1) = \left[\underline{A}(\underline{\bar{a}}) + \underline{B}(\underline{\bar{a}})\underline{F}_y\underline{C}\right]\underline{x}(k) + \underline{B}(\underline{\bar{a}})\underline{v}(k) + \underline{B}(\underline{\bar{a}})\underline{F}_y\underline{\bar{w}}(k) \tag{9}$$

$$\underline{y}(k) = \underline{Cx}(k) + \underline{\bar{w}}(k) \tag{7}$$

The noise term $\underline{\bar{w}}(k)$ is an additional uncertainty beyond that

of $\bar{\underline{a}}$ itself. To identify $\bar{\underline{a}}$ with noise in the measurements, it is desirable to have the output $\underline{y}(k)$ contain as much information about $\bar{\underline{a}}$ as possible; that is, the weighted trace of $\underline{M}$ for this problem should be as large as possible. It is this case, measurement noise only, that is addressed in this dissertation.

If measurement and process noise are added to the model then Eqs (3) and (4) become:

$$\underline{x}(k+1) = \underline{A}(\bar{\underline{a}})\underline{x}(k) + \underline{B}(\bar{\underline{a}})\underline{u}(k) + \underline{\phi}(k) \tag{10}$$

$$\underline{y}(k) = \underline{Cx}(k) + \bar{\underline{w}}(k) \tag{11}$$

Maximizing a function of $\underline{M}$ is meaningful in this case, but calculating $\underline{M}$ becomes very difficult. Mehra (Ref 21) shows that calculating the conditional likelihood function $\ln p(Y_N|\underline{a})$ for this model is given in terms of the Kalman filter model corresponding to Eqs (10) and (11). The Kalman filter model, as presented in Mehra's paper, is

$$\hat{\underline{x}}(k+1) = \underline{A}(\bar{\underline{a}}_o)\hat{\underline{x}}(k) + \underline{B}(\bar{\underline{a}}_o)\underline{u}(k) + \bar{\underline{K}}(k)\underline{\nu}(k) \tag{12}$$

$$\underline{y}(k) = \underline{C}\hat{\underline{x}}(k) + \underline{\Sigma}(k)\underline{\nu}(k) \tag{13}$$

where

$$\hat{\underline{x}}(k) = E\left[\underline{x}(k)\middle|\,\underline{y}(1), \underline{y}(2), \cdots, \underline{y}(k-1)\right]$$

$$\underline{\nu}(k) = \left[\underline{CP}(k)^T\underline{C}^T + \underline{R}\right]^{-\frac{1}{2}}(\underline{y}(k) - \underline{C}\hat{\underline{x}}(k))$$

$$\bar{\underline{K}}(k) = \underline{A}(\bar{\underline{a}}_o)\underline{P}(k)\underline{C}^T\left[\underline{CP}(k)\underline{C}^T + \underline{R}\right]^{-\frac{1}{2}}$$

$$\underline{P}(k+1) = \underline{A}(\bar{\underline{a}}_o)\underline{P}(k)\underline{A}^T(\bar{\underline{a}}_o) - \bar{\underline{K}}(k)\bar{\underline{K}}^T(k) + \underline{Q}$$

$$\underline{\Sigma}(k) = \left[\underline{CP}(k)\underline{C}^T + \underline{R}\right]^{\frac{1}{2}}$$

The measurement and process noises are assumed to be white and Gaussian, with

$$E\left[\underline{\phi}(k)\right] = \underline{0} \quad ; \quad E\left[\underline{\bar{w}}(k)\right] = \underline{0} \quad ; \quad E\left[\underline{\nu}(k)\right] = \underline{0}$$

$$E\left[\underline{\phi}(k)\underline{\phi}^T(j)\right] = \underline{Q}\,\delta_{kj}$$

$$E\left[\underline{\bar{w}}(k)\underline{\bar{w}}^T(j)\right] = \underline{R}\,\delta_{kj}$$

$$E\left[\underline{\bar{w}}(k)\underline{\phi}^T(j)\right] = \underline{0}$$

$$E\left[\underline{x}(0)\underline{\bar{w}}^T(k)\right] = \underline{0}$$

$$E\left[\underline{x}(0)\underline{\phi}^T(k)\right] = \underline{0}$$

$$E\left[\underline{\nu}(k)\underline{\nu}^T(j)\right] = \underline{I}\,\delta_{kj}$$

As shown by Mehra, the main reason for the complexity of $\underline{M}$ is that the covariance matrix of $\underline{\Sigma}(k)\underline{\nu}(k)$ is time varying and dependent on $\underline{a}$ in a nonlinear manner. Taking the partials of this matrix with respect to $\underline{\bar{a}}$ becomes difficult.

Once $\underline{M}$ is calculated, it is shown in the next chapter that $\partial \underline{M}/\partial \underline{\bar{a}}$ has to be evaluated for the gradient algorithm. Taking the partial of $\underline{M}$ with respect to $\underline{\tilde{a}}$ increases the complexity of the problem significantly. Solving the problem of determining the optimal control for this model with process noise is considered beyond the scope of this research and is not addressed. The work developed in this research should provide insight into solving this more difficult, but practically significant, problem.

Objective and Organization

The objective of this research is to develop an algorithm that will calculate the constant feedback matrix and open-loop control term that optimally aids in identifying a

set of unknown parameters in a given linear system model. The constraints on the system are an energy constraint on the open-loop portion of the controls and a restriction that poles of the feedback system must remain within a predetermined constraint space. This space depends on the stability and responsiveness required of the system.

To achieve this objective, an algorithm is developed in Chapter II that considers the case of a system with a single input control and a single unknown parameter. This reduces the complexity of the problem and makes it easier to understand the concepts and procedure used to develop the algorithm.

To help in understanding the solution, a three dimensional problem is selected, and the optimal control is calculated. This is performed in Chapter III. Although there is only one unknown parameter in the system, its influence is dispersed throughout the system and control matrices in a nonlinear manner. Cases with both two and three dimensional output vectors are studied.

An estimator is also designed to demonstrate enhanced parameter estimation with use of the affine control. The calculated optimal affine control and a noise generator are applied to the example case to generate system outputs. The estimator then computes the estimate of the unknown parameter by using the output measurements and input controls. The results of the estimator for different noise levels and

different number of time sequences are presented.

In Chapter IV extensions to the problem addressed in Chapter II are presented. For these extensions it can be easily seen by the complexity of the algorithms that the computational burden increases substantially. Chapter V concludes by summarizing the contributions made in this research along with recommendations for future research in this area.

II Optimal Feedback Control Design

Introduction

Consider a discrete system assumed to be modelled adequately by

$$\underline{x}(k+1) = \underline{A}(\bar{a})\underline{x}(k) + \underline{B}(\bar{a})u(k) \tag{14}$$

$$\underline{y}(k) = \underline{C}\underline{x}(k) + \bar{\underline{w}}(k) \tag{15}$$

where $\underline{x}(k)$ is an n-dimensional state vector, u(k) is a scalar control, $\underline{y}(k)$ is an r-dimensional output vector, and $\bar{\underline{w}}(k)$ is an r-dimensional measurement noise vector. $\underline{A}(\bar{a})$ is an nxn time-invariant plant matrix that may have confluent eigenvalues (i.e. nondistinct), $\underline{B}(\bar{a})$ is an nx1 time-invariant control matrix, and $\underline{C}$ is an rxn time-invariant output matrix that is independent of the parameter $\bar{a}$. The effects of unknown scalar parameter $\bar{a}$ are confined to $\underline{A}(\bar{a})$ and $\underline{B}(\bar{a})$ only, and $\bar{a}_o$ is a nominal value of $\bar{a}$. Vector $\bar{\underline{w}}(k)$ is a Gaussian white noise sequence with

$$E\,\bar{\underline{w}}(k) = \underline{0} \quad ; \quad E\,\bar{\underline{w}}(k)\bar{\underline{w}}^T(j) = \underline{R}\,\delta_{kj} \tag{16}$$

The initial condition $\underline{x}(0)$ is assumed to be known exactly. It is also assumed that the system described by Eqs (14) and (15) is completely controllable and completely observable for all $\bar{a}$, i.e.,

$$\text{rank}\left[\underline{B}(\bar{a}),\ \underline{A}(\bar{a})\underline{B}(\bar{a}),\cdots,\ \underline{A}^{n-1}(\bar{a})\underline{B}(\bar{a})\right] = n$$

$$\text{rank}\left[\underline{C}^T,\ \underline{A}^T(\bar{a})\underline{C}^T,\cdots,(\underline{A}^{n-1}(\bar{a}))^T\underline{C}^T\right] = n$$

Without this assumption there may be modes of the system that cannot be controlled to the desired value and the unknown parameter may not be observable in the output.

The control u(k) is restricted to having a linear output feedback term and an open-loop external control term, a member of the affine control law. Thus,

$$u(k) = \underline{F}_y \underline{y}(k) + v(k) \tag{17}$$

where $\underline{F}_y$ is an 1xr time-invariant feedback matrix and v(k) is a scalar control. An energy constraint is imposed on the external control term v(k) such that

$$\sum_{k=0}^{N-1} v^2(k) \leq E \tag{18}$$

where N-1 is the last instant of time the control is applied, i.e. N = final time instant. Without a constraint on v(k) it would be desirable to have as large a signal as possible in order to get an output with a large signal to noise ratio. Practically this would not be possible since there is a limited amount of energy available to control most systems. The restriction on the feedback matrix $\underline{F}_y$ is that the eigenvalues of the feedback homogeneous system

$$\underline{x}(k+1) = \left[\underline{A}(\bar{a}) + \underline{B}(\bar{a})\underline{F}_y\underline{C}\right]\underline{x}(k) \tag{19}$$

be members of a constraint space Φ that is predetermined so as to ensure desirable stability and time response characteristics (see page 53). Therefore, the objective of the

research is to determine the optimal feedback matrix $\underline{F}_y$ and optimal external control sequence vector $\underline{V}$, where this vector is $\underline{V}^T = [v(0), v(1), \cdots, v(N-1)]$ such that the best estimate of the unknown parameter can be found while maintaining the constraint requirements.

It is assumed that the estimator that will be used to identify $\bar{a}$ will be in the class of efficient or asymptotically efficient estimators. That is, they are asymptotically unbiased, minimum variance estimators such as maximum likelihood estimators. Using this assumption, the derivation of the optimal $\underline{F}_y$ and $\underline{V}$ can be separated from the derivation of the estimator (Ref 23). Maximization of the Fisher information matrix is the criterion used to calculate $\underline{F}_y$ and $\underline{V}$. For the case of the single unknown parameter, this matrix becomes a scalar. For any unbiased estimate $\hat{\bar{a}}$ of the unknown parameter $\bar{a}$, it is known that a lower bound of its error variance is given by the Cramér-Rao inequality (Ref 23)

$$E\left[(\bar{a}-\hat{\bar{a}})^2 \middle| \bar{a}\right] \geq M^{-1} \tag{20}$$

where $\bar{a}$ is such that

$$E\left[\hat{\bar{a}} \middle| \bar{a}\right] = \bar{a} \tag{21}$$

The Fisher information scalar is given by

$$M = \underset{Y_N}{E}\left[\left(\frac{\partial \ln p(Y_N|\bar{a})}{\partial \bar{a}}\right)^2\right] \tag{22}$$

where $Y_N = [\underline{y}(k): k = 0, 1, \cdots, N]$. Throughout this report all partials are evaluated at $\bar{a} = \bar{a}_o$.

For an asymptotically efficient estimator the lower bound is asymptotically achieved; therefore, by maximizing the information scalar, the estimation error is decreased. The $\underline{F}_y$ and $\underline{V}$ are found that maximize M.

Information Criterion

Substituting Eq (17) into Eq (14), the equations for the system become:

$$\underline{x}(k+1) = [\underline{A}(\bar{a}) + \underline{B}(\bar{a})\underline{F}_y\underline{C}]\,\underline{x}(k) + \underline{B}(\bar{a})v(k) + \underline{B}(\bar{a})\underline{F}_y\underline{\bar{w}}(k) \quad (23)$$

$$\underline{y}(k) = \underline{C}\underline{x}(k) + \underline{\bar{w}}(k) \quad (24)$$

The feedback system has both process and measurement noise, but they are the same. It is shown that this simplifies the means of calculating the information scalar.

In order to solve M, $p(Y_N|\bar{a})$ must be calculated. Now, by repeated application of Bayes' Rule,

$$\begin{aligned} p(Y_N|\bar{a}) &= p(\underline{y}(N)|Y_{N-1},\bar{a})p(Y_{N-1}|\bar{a}) \\ &= p(\underline{y}(N)|Y_{N-1},\bar{a})p(\underline{y}(N-1)|Y_{N-2},\bar{a})p(Y_{N-2}|\bar{a}) \\ &= \prod_{j=1}^{N} p(\underline{y}(j)|Y_{j-1},\bar{a}) \end{aligned} \quad (25)$$

From Eq (24) if $\underline{x}(0)$ and $\underline{y}(0)$ are known, $\underline{\bar{w}}(0)$ can be computed. From Eq (23) if $\underline{\bar{w}}(0)$, $\underline{x}(0)$, and $\bar{a}$ are known, $\underline{x}(1)$ can be computed. This process can be repeated so that if

$\underline{x}(0)$, $\bar{a}$, and Y_{j-1} are known, then $W_{j-1} = \left[\underline{\bar{w}}(k)\text{: } k = 0, 1, \cdots, j-1\right]$ is known and so is $\underline{x}(j)$. With this information the following can be written:

$$p(\underline{y}(j)|\, Y_{j-1}, \bar{a}) = p(\underline{y}(j)|\, W_{j-1}, \bar{a})$$

$$= p(\underline{y}(j)|\, \underline{x}(j)) \tag{26}$$

$$E\left[\underline{y}(j)|\, \underline{x}(j)\right] = \underline{Cx}(j) \tag{27}$$

$$E\left[(\underline{y}(j) - \underline{Cx}(j))(\underline{y}(j) - \underline{Cx}(j))^T\right] = E\left[\underline{\bar{w}}(j)\underline{\bar{w}}^T(j)\right]$$

$$= \underline{R} \tag{28}$$

Therefore,

$$p(\underline{y}(j)|\, Y_{j-1}, \bar{a}) = \frac{1}{\sqrt{2\pi|\underline{R}|}}\, e^{-\frac{1}{2}\left[\underline{\bar{n}}^T(j)\underline{R}^{-1}\underline{\bar{n}}(j)\right]} \tag{29}$$

where $\underline{\bar{n}}(j) = \underline{y}(j) - \underline{Cx}(j)$.

Now

$$p(Y_N|\, \bar{a}) = \prod_{j=1}^{N} \frac{1}{\sqrt{2\pi|\underline{R}|}}\, e^{-\frac{1}{2}\left[\underline{\bar{n}}^T(j)\underline{R}^{-1}\underline{\bar{n}}(j)\right]} \tag{30}$$

and

$$\ln p(Y_N|\, \bar{a}) = \text{Constant} - \tfrac{1}{2}\sum_{j=1}^{N} \underline{\bar{n}}^T(j)\underline{R}^{-1}\underline{\bar{n}}(j) \tag{31}$$

Therefore

$$\frac{\partial \ln p(Y_N|\, \bar{a})}{\partial\, \bar{a}} = -\sum_{j=1}^{N} \frac{\partial\, \underline{\bar{n}}^T(j)}{\partial\, \bar{a}}\, \underline{R}^{-1}\underline{\bar{n}}(j) \tag{32}$$

and

$$\left[\frac{\partial \ln p(Y_N|\bar{a})}{\partial \bar{a}}\right]^2 = \sum_{j=1}^{N}\sum_{k=1}^{N}\frac{\partial \bar{\underline{n}}^T(j)}{\partial \bar{a}}\underline{R}^{-1}\bar{\underline{n}}(j)\bar{\underline{n}}^T(k)\underline{R}^{-1}\frac{\partial \bar{\underline{n}}(k)}{\partial \bar{a}}$$

$$= \sum_{j=1}^{N}\frac{\partial \bar{\underline{n}}^T(j)}{\partial \bar{a}}\underline{R}^{-1}\bar{\underline{n}}(j)\bar{\underline{n}}^T(j)\underline{R}^{-1}\frac{\partial \bar{\underline{n}}(j)}{\partial \bar{a}}$$

$$+ 2\sum_{j=1}^{N-1}\sum_{k=j+1}^{N}\frac{\partial \bar{\underline{n}}^T(j)}{\partial \bar{a}}\underline{R}^{-1}\bar{\underline{n}}(j)\bar{\underline{n}}^T(k)\underline{R}^{-1}\frac{\partial \bar{\underline{n}}(k)}{\partial \bar{a}} \qquad (33)$$

To obtain the information scalar, the expected value of Eq (33) is computed. The expected value of the second term on the right side of Eq (33) is zero. This can be seen from the following equations:

$$\bar{\underline{n}}(j) = \underline{y}(j) - \underline{C}\underline{x}(j) = \bar{\underline{w}}(j) \qquad (34)$$

$$\frac{\partial \bar{\underline{n}}(j)}{\partial \bar{a}} = \frac{\partial}{\partial \bar{a}}\left[\underline{y}(j) - \underline{C}\underline{x}(j)\right] = -\underline{C}\frac{\partial \underline{x}(j)}{\partial \bar{a}} \qquad (35)$$

A term of the double summation is

$$2\frac{\partial \bar{\underline{n}}^T(j)}{\partial \bar{a}}\underline{R}^{-1}\bar{\underline{n}}(j)\bar{\underline{n}}^T(k)\underline{R}^{-1}\frac{\partial \bar{\underline{n}}(k)}{\partial \bar{a}} \qquad \text{where } k > j$$

$$= 2\frac{\partial \underline{x}^T(j)}{\partial \bar{a}}\underline{C}^T\underline{R}^{-1}\bar{\underline{w}}(j)\bar{\underline{w}}^T(k)\underline{R}^{-1}\underline{C}\frac{\partial \underline{x}(k)}{\partial \bar{a}}$$

$$= 2\bar{\underline{w}}^T(k)\underline{R}^{-1}\underline{C}\frac{\partial \underline{x}(k)}{\partial \bar{a}}\,\frac{\partial \underline{x}^T(j)}{\partial \bar{a}}\underline{C}^T\underline{R}^{-1}\bar{\underline{w}}(j) \qquad (36)$$

From Eq (23)

$$\frac{\partial \underline{x}(k)}{\partial \bar{a}} = \left[\underline{A}(\bar{a}) + \underline{B}(\bar{a})\underline{F}_y\underline{C}\right] \frac{\partial \underline{x}(k-1)}{\partial \bar{a}}$$

$$+ \frac{\partial}{\partial \bar{a}}\left[\underline{A}(\bar{a}) + \underline{B}(\bar{a})\underline{F}_y\underline{C}\right] \underline{x}(k-1)$$

$$+ \frac{\partial \underline{B}(\bar{a})}{\partial \bar{a}} v(k-1) + \frac{\partial \underline{B}(\bar{a})}{\partial \bar{a}} \underline{F}_y \bar{\underline{w}}(k-1) \qquad (37)$$

It can be seen that $\partial \underline{x}(k)/\partial \bar{a}$ is a function of $\bar{\underline{w}}(k-1)$, $\bar{\underline{w}}(k-2), \cdots, \bar{\underline{w}}(j), \cdots, \bar{\underline{w}}(0)$ and that $\partial \underline{x}(j)/\partial \bar{a}$ is a function of $\bar{\underline{w}}(j-1), \bar{\underline{w}}(j-2), \cdots, \bar{\underline{w}}(0)$. The expected value of Eq (36) is

$$E\left[2\bar{\underline{w}}^T(k)\underline{R}^{-1}\underline{C}\frac{\partial \underline{x}(k)}{\partial \bar{a}} \frac{\partial \underline{x}^T(j)}{\partial \bar{a}}\underline{C}^T\underline{R}^{-1}\bar{\underline{w}}(j)\right]$$

$$= 2E\left[\bar{\underline{w}}^T(k)\right] E\left[\underline{R}^{-1}\underline{C}\frac{\partial \underline{x}(k)}{\partial \bar{a}} \frac{\partial \underline{x}^T(j)}{\partial \bar{a}}\underline{C}^T\underline{R}^{-1}\bar{\underline{w}}(j)\right] \qquad (38)$$

because $\bar{\underline{w}}(k)$ is independent of the other terms. Since $E\left[\bar{\underline{w}}(k)\right] = \underline{0}$, Eq (38) is zero.

The information scalar then becomes:

$$M = E\left[\sum_{j=1}^{N} \frac{\partial \bar{\underline{n}}^T(j)}{\partial \bar{a}}\underline{R}^{-1}\bar{\underline{n}}(j)\bar{\underline{n}}^T(j)\underline{R}^{-1}\frac{\partial \bar{\underline{n}}(j)}{\partial \bar{a}}\right] \qquad (39)$$

The summation is a scalar and using the fact that the trace of a scalar equals the scalar and

$$\mathrm{tr}\left[\underline{XYZ}\right] = \mathrm{tr}\left[\underline{ZXY}\right] = \mathrm{tr}\left[\underline{YZX}\right]$$

where $\underline{X}$, $\underline{Y}$, and $\underline{Z}$ are matrices, we obtain, after substituting Eqs (34) and (35) into Eq (39)

$$M = E\left[\sum_{j=1}^{N} \mathrm{tr}\left[\frac{\partial \underline{x}(j)}{\partial \bar{a}} \frac{\partial \underline{x}^T(j)}{\partial \bar{a}} \underline{C}^T \underline{R}^{-1} \bar{\underline{w}}(j) \bar{\underline{w}}^T(j) \underline{R}^{-1} \underline{C}\right]\right] \tag{40}$$

Since $E\left[\sum_{j=1}^{N} \mathrm{tr}\underline{X}_j\right] = \sum_{j=1}^{N} \mathrm{tr}E\left[X_j\right]$ and it was shown that

$\partial \underline{x}(j)/\partial \bar{a}$ is independent of $\bar{\underline{w}}(j)$, then

$$M = \sum_{j=1}^{N} \mathrm{tr}E\left[\frac{\partial \underline{x}(j)}{\partial \bar{a}} \frac{\partial \underline{x}^T(j)}{\partial \bar{a}} \underline{C}^T \underline{R}^{-1} \bar{\underline{w}}(j) \bar{\underline{w}}^T(j) \underline{R}^{-1} \underline{C}\right]$$

$$= \sum_{j=1}^{N} \mathrm{tr}\left[E\left[\frac{\partial \underline{x}(j)}{\partial \bar{a}} \frac{\partial \underline{x}^T(j)}{\partial \bar{a}}\right] \underline{C}^T \underline{R}^{-1} E\left[\bar{\underline{w}}(j) \bar{\underline{w}}^T(j)\right] \underline{R}^{-1} \underline{C}\right] \tag{41}$$

This is valid because $\left[\partial \underline{x}(j)/\partial \bar{a}\right]\left[\partial \underline{x}^T(j)/\partial \bar{a}\right]$ is independent of $\bar{\underline{w}}(j)\bar{\underline{w}}^T(j)$ and the expected value of the product of two independent variables is equal to the product of the expected values of the variables. Equation (41) becomes

$$M = \sum_{j=1}^{N} \mathrm{tr}\left[E\left[\frac{\partial \underline{x}(j)}{\partial \bar{a}} \frac{\partial \underline{x}^T(j)}{\partial \bar{a}}\right] \underline{C}^T \underline{R}^{-1} \underline{C}\right] \tag{42}$$

because $E\left[\bar{\underline{w}}(j)\bar{\underline{w}}^T(j)\right] = \underline{R}$.

Combining Eqs (23) and (24) with Eq (37) to form an augmented system, the equations become:

$$\begin{bmatrix} \underline{x}(k+1) \\ \hline \dfrac{\partial\, \underline{x}(k+1)}{\partial\, \bar{a}} \end{bmatrix} = \left[\begin{array}{c|c} \underline{A}(\bar{a}) + \underline{B}(\bar{a})\underline{F}_y\underline{C} & \underline{O} \\ \hline \dfrac{\partial \underline{A}(\bar{a})}{\partial\, \bar{a}} + \dfrac{\partial\, \underline{B}(\bar{a})}{\partial\, \bar{a}}\underline{F}_y\underline{C} & \underline{A}(\bar{a}) + \underline{B}(\bar{a})\underline{F}_y\underline{C} \end{array}\right]$$

$$X \begin{bmatrix} \underline{x}(k) \\ \hline \dfrac{\partial\, \underline{x}(k)}{\partial\, \bar{a}} \end{bmatrix} + \begin{bmatrix} \underline{B}(\bar{a}) \\ \hline \dfrac{\partial\, \underline{B}(\bar{a})}{\partial\, \bar{a}} \end{bmatrix} v(k) + \begin{bmatrix} \underline{B}(\bar{a})\underline{F}_y \\ \hline \dfrac{\partial \underline{B}(\bar{a})}{\partial\, \bar{a}}\underline{F}_y \end{bmatrix} \bar{\underline{w}}(k) \tag{43}$$

Let

$$\underline{X}_A^T(k+1) = \left[\underline{x}^T(k+1),\ \frac{\partial\, \underline{x}^T(k+1)}{\partial\, \bar{a}}\right]_{1x2n} \tag{44}$$

$$\underline{B}_A^T = \left[\underline{B}^T(\bar{a}),\ \frac{\partial \underline{B}^T(\bar{a})}{\partial\, \bar{a}}\right]_{1x2n} \tag{45}$$

$$\underline{D}_A^T = \left[\underline{F}_y{}^T\underline{B}^T(\bar{a}),\ \underline{F}_y{}^T\frac{\partial\, \underline{B}^T(\bar{a})}{\partial\, \bar{a}}\right]_{rx2n} = \underline{F}_y{}^T\underline{B}_A^T \tag{46}$$

and

$$\underline{A}_A = \left[\begin{array}{c|c} \underline{A}(\bar{a}) + \underline{B}(\bar{a})\underline{F}_y\underline{C} & \underline{O} \\ \hline \dfrac{\partial\, \underline{A}(\bar{a})}{\partial\, \bar{a}} + \dfrac{\partial \underline{B}(\bar{a})}{\partial\, \bar{a}}\underline{F}_y\underline{C} & \underline{A}(a) + \underline{B}(a)\underline{F}_y\underline{C} \end{array}\right]_{2nx2n} \tag{47}$$

Using this notation, Eq (43) becomes:

$$\underline{X}_A(k+1) = \underline{A}_A\underline{X}_A(k) + \underline{B}_A(k)v(k) + \underline{D}_A\bar{\underline{w}}(k) \tag{48}$$

Since an expression in Eq (42) is $E\left[\dfrac{\partial\, \underline{x}(j)}{\partial\, \bar{a}}\ \dfrac{\partial\, \underline{x}^T(j)}{\partial\, \bar{a}}\right]$, it is necessary to compute this expected value. To get this let

$$E\left[\underline{X}_A(k+1)\right] = \hat{\underline{X}}_A(k+1)$$

$$= \underline{A}_A \hat{\underline{X}}_A(k) + \underline{B}_A v(k) \tag{49}$$

then

$$E\left[\frac{\partial \underline{x}(k)}{\partial \bar{a}} \; \frac{\partial \underline{x}^T(k)}{\partial \bar{a}}\right] = E\left[\underline{h}^T \underline{X}_A(k) \underline{X}_A^T(k) \underline{h}\right]$$

$$= \underline{h}^T \left[\hat{\underline{X}}_A(k) \hat{\underline{X}}_A^T(k) + \underline{P}_A(k)\right] \underline{h} \tag{50}$$

where

$$h^T = \left[\underline{O} \;\vdots\; \underline{I}\right]_{nx2n} \tag{51}$$

$\underline{I}$ is an nxn identity matrix

$$\underline{P}_A(k) = \underline{A}_A \underline{P}_A(k-1) \underline{A}_A^T + \underline{D}_A \underline{R} \underline{D}_A^T \tag{52}$$

$\underline{P}_A(0) = \underline{O}$ since $\underline{x}(0)$ and $\partial \underline{x}(0)/\partial \bar{a} = \underline{O}$ and are known exactly.

Now

$$M = \sum_{j=1}^{N} \mathrm{tr}\left[\underline{h}^T \left[\hat{\underline{X}}_A(j) \hat{\underline{X}}_A^T(j) + \underline{P}_A(j)\right] \underline{hC}^T \underline{R}^{-1} \underline{C}\right]$$

$$= \sum_{j=1}^{N} \mathrm{tr}\left[\underline{h}^T \hat{\underline{X}}_A(j) \hat{\underline{X}}_A^T(j) \underline{hC}^T \underline{R}^{-1} \underline{C}\right]$$

$$+ \sum_{j=1}^{N} \mathrm{tr}\left[\underline{h}^T \underline{P}_A(j) \underline{hC}^T \underline{R}^{-1} \underline{C}\right] \tag{53}$$

The term inside the brackets in the first summation is a scalar. Using the properties of a trace, M becomes

$$M = \sum_{j=1}^{N} \hat{\underline{X}}_A^T(j) \underline{hC}^T \underline{R}^{-1} \underline{Ch}^T \hat{\underline{X}}_A(j) + \sum_{j=1}^{N} \mathrm{tr}\left[\underline{h}^T \underline{P}_A(j) \underline{hC}^T \underline{R}^{-1} \underline{C}\right] \tag{54}$$

If there were no feedback, then the information scalar as given by Mehra (Ref 20) would be:

$$M = \sum_{j=1}^{N} \underline{\hat{X}}_A^T(j)\underline{hC}^T\underline{R}^{-1}\underline{Ch}^T\underline{\hat{X}}_A(j) \tag{55}$$

The differences between Eqs (54) and (55) are that in Eq (54) the vector $\underline{\hat{X}}_A(j)$ is a function of $\underline{F}_y$ while in Eq (55) it is not, and that Eq (54) includes the additional term which is called $J(\underline{F}_y)$. This additional term

$$J(\underline{F}_y) = \sum_{j=1}^{N} tr\left[\underline{h}^T\underline{P}_A(j)\underline{hC}^T\underline{R}^{-1}\underline{C}\right] \tag{56}$$

represents the weighted trace of the error covariance for the estimate of the vector $\partial\,\underline{x}(j)/\partial\bar{a}$ where the weighting matrix is $\underline{C}^T\underline{R}^{-1}\underline{C}$.

From Eq (52) it can be easily shown that

$$\underline{P}_A(k) = \sum_{j=1}^{k} \underline{A}_A^{l-1}\underline{D}_A\underline{RD}_A^T(\underline{A}_A^T)^{l-1} \tag{57}$$

$$\underline{P}_A(0) = \underline{0} \tag{58}$$

From Eq (56) it can be seen that for a large uncertainty about how a change in the parameter $\bar{a}$ affects a change in the state $\underline{x}(j)$, i.e. large $\underline{P}_A(k)$, $J(\underline{F}_y)$ is increased. We are dealing with $\underline{h}^T\underline{P}_A(k)\underline{h}$, the lower right partition of the matrix $\underline{P}_A(k)$ and with $\partial\underline{x}(k)/\partial\bar{a}$. This means that more information about $\bar{a}$ is gained by feeding back the output if $J(\underline{F}_y)$ is always positive which is true and proven in Appen-

dix A. If the uncertainty is small, little additional information is gained by feedback. From Eqs (56) and (57) it can be seen that measurement noise increases the error covariance matrix but decreases the weighting matrix. This implies that the magnitude of the measurement noise may have a small effect on this additional term. For example, consider the case where $\underline{R} = r\underline{I}$. The term $\underline{C}^T\underline{R}^{-1}\underline{C}$ becomes $\frac{1}{r}\underline{C}^T\underline{C}$ and Eq (57) becomes:

$$\underline{P}_A(k) = r \sum_{j=1}^{k} \underline{A}_A^{l-1} \underline{D}_A \underline{D}_A^T (\underline{A}_A^T)^{l-1} \tag{59}$$

Therefore from Eq (56) it is seen that the r terms cancel and that $J(\underline{F}_y)$ is independent of the measurement noise.

To see the effect of the feedback vector on $J(\underline{F}_y)$, Eq (57) can be expanded to

$$\underline{P}_A(j) = \sum_{l=1}^{j} \underline{A}_A^{l-1} \underline{B}_A \underline{F}_y \underline{R} \underline{F}_y{}^T \underline{B}_A^T (\underline{A}_A^T)^{l-1}$$

$$= \underline{F}_y \underline{R} \underline{F}_y{}^T \sum_{l=1}^{j} \underline{A}_A^{l-1} \underline{B}_A \underline{B}_A^T (\underline{A}_A^T)^{l-1} \tag{60}$$

since $\underline{F}_y \underline{R} \underline{F}_y{}^T$ is a scalar. From Eq (56)

$$J(\underline{F}_y) = \sum_{j=1}^{N} \mathrm{tr}\left[\underline{h}^T \underline{F}_y \underline{R} \underline{F}_y{}^T \sum_{l=1}^{j} \underline{A}_A^{l-1} \underline{B}_A \underline{B}_A^T (\underline{A}_A^T)^{l-1} \underline{h} \underline{C}^T \underline{R}^{-1} \underline{C}\right]$$

$$= \sum_{j=1}^{N} j \underline{F}_y \underline{R} \underline{F}_y{}^T \mathrm{tr}\left[\underline{h}^T \underline{A}_A^{N-j} \underline{B}_A \underline{B}_A^T (\underline{A}_A^T)^{N-j} \underline{h} \underline{C}^T \underline{R}^{-1} \underline{C}\right] \tag{61}$$

because

$$\sum_{j=1}^{N} \sum_{l=1}^{j} \underline{X}^{l-1} = \sum_{j=1}^{N} j\underline{X}^{N-j} \tag{62}$$

and

$$\mathrm{tr} \sum_{j=1}^{k} \underline{X}(j) = \sum_{j=1}^{k} \mathrm{tr}\underline{X}(j) \tag{63}$$

From Eq (61) it is easily seen that if there is no feedback, $\underline{F}_y = \underline{0}$, then $J(\underline{F}_y) = 0$. Another important fact about $J(\underline{F}_y)$ is that this term can never be negative (proved in Appendix A). This implies that the additional term can never decrease information about the parameter $\bar{a}$ because of feedback.

The expressions for the information scalar and the constraints are known. It is now necessary to apply an optimization technique to maximize the information available. This is done by using a gradient technique.

Optimizing the Information

The optimal $\underline{F}_y$ must be such that it maximizes the sum of the two terms in Eq (54). To find the $\underline{V}$ and $\underline{F}_y$ that maximize M, a gradient method is used. The reason a first order method is selected is that the complexity increases significantly if higher order partial derivatives are required. This will become more obvious subsequently when the first order partials are derived. Another advantage is that

the gradient method will always converge at least to a local maximum, whereas some higher order methods only converge if the initial conditions are close to the optimum value. Disadvantages are that the global maximum may not be obtained and that convergence is usually slow near the optimum value. In using the gradient method, the gradient

$$\left.\frac{\partial M}{\partial \begin{bmatrix} \underline{F}_y^T \\ \underline{V} \end{bmatrix}}\right|_{\substack{\underline{F}_y^T = \underline{F}_{yo}^T \\ \underline{V} = \underline{V}_o}}$$

has to be evaluated. This would require taking the partial of a scalar with respect to an (r+N) dimensional vector (see Appendix B for an explanation of this procedure).

Evaluation of this partial at each iteration step would consume a large amount of computation time. It is shown that only the r-dimensional vector

$$\left.\frac{\partial M}{\partial \underline{F}_y^T}\right|_{\substack{\underline{F}_y^T = \underline{F}_{yo}^T \\ \underline{V} = \underline{V}_o}}$$

has to be calculated. In order to do this, the expression for M can be expanded so as to simplify the partials.

<u>Modified Form of M</u>. The first term of Eq (54) for M can be written again as

$$K = \sum_{k=1}^{N} \hat{\underline{X}}_A^T(k)\underline{hC}^T\underline{R}^{-1}\underline{Ch}^T\hat{\underline{X}}_A(k) \qquad (64)$$

and from Eq (49)

$$
\begin{aligned}
\hat{\underline{X}}_A(k) &= \underline{A}_A\hat{\underline{X}}_A(k-1) + \underline{B}_A v(k-1) \\
&= \underline{A}_A\left[\underline{A}_A\hat{\underline{X}}_A(k-2) + \underline{B}_A v(k-2)\right] + \underline{B}_A v(k-1) \\
&\bullet \\
&\bullet \\
&\bullet \\
&= \underline{A}_A^k\hat{\underline{X}}_A(0) + \underline{A}_A^{k-1}\underline{B}_A v(0) + \underline{A}_A^{k-2}\underline{B}_A v(1) \\
&\qquad + \dots + \underline{A}_A^0\underline{B}_A v(k-1) \\
&= \left[\underline{A}_A^k,\ \underline{A}_A^{k-1}B_A, \dots, \underline{A}_A^0\ B_A\right]_{2mx(2n+k)} \begin{bmatrix} \hat{\underline{X}}_A(0) \\ v(0) \\ \bullet \\ \bullet \\ \bullet \\ v(k-1) \end{bmatrix}_{(2n+k)x1}
\end{aligned}
\tag{65}
$$

where $\hat{\underline{X}}_A^T(0) = \left[\underline{x}^T(0),\ \underline{0}_{1xn}\right]_{1x2n}$.

Let

$$
\underline{A}_F(\bar{a}) = \underline{A}(\bar{a}) + \underline{B}(\bar{a})\underline{F}_y\underline{C} \tag{66}
$$

where the subscript F is used to describe the plant matrix for the feedback system. Then

$$
\underline{A}_A = \left[\begin{array}{c|c} \underline{A}_F(\bar{a}) & \underline{0} \\ \hline \dfrac{\partial\ \underline{A}_F(\bar{a})}{\partial\ \bar{a}} & \underline{A}_F(\bar{a}) \end{array}\right] \tag{67}
$$

and

$$\underline{A}_A^k = \left[\begin{array}{c|c} \underline{A}_F^k(\bar{a}) & \underline{O} \\ \hline \dfrac{\partial \underline{A}_F^k(\bar{a})}{\partial \bar{a}} & \underline{A}_F^k(\bar{a}) \end{array}\right] \tag{68}$$

Proof of this is shown in Appendix C.

Also

$$\underline{A}_A^k \underline{B}_A = \left[\begin{array}{c} \underline{A}_F^k(\bar{a})\underline{B}(\bar{a}) \\ \hline \dfrac{\partial}{\partial \bar{a}}\left[\underline{A}_F^k(\bar{a})\underline{B}(\bar{a})\right] \end{array}\right] \tag{69}$$

because the bottom term of this matrix can be written as

$$\frac{\partial}{\partial \bar{a}}\left[\underline{A}_F^k(\bar{a})\underline{B}(\bar{a})\right] = \frac{\partial\, \underline{A}_F^k(\bar{a})}{\partial \bar{a}}\underline{B}(\bar{a}) + \underline{A}_F^k(\bar{a})\frac{\partial\, \underline{B}(\bar{a})}{\partial \bar{a}} \tag{70}$$

and it is the right hand terms that are obtained when $\underline{A}_A^k$ is postmultiplied by $\underline{B}_A$.

Let

$$\underline{V}_{X_A}^T(k) = \left[\hat{\underline{X}}_A^T(0),\ v(0),\ v(1), \cdots,\ v(k-1)\right] \tag{71}$$

Elements n+1 through 2n of $\underline{V}_{X_A}(k)$ equal zero because of the equation $\partial\, \underline{x}(0)/\partial \bar{a} = \underline{O}$ so Eq (65) becomes

$$\hat{\underline{X}}_A(k) = \left[\begin{array}{c|c|c|c} \underline{A}_F^k(\bar{a}) & \underline{A}_F^{k-1}(\bar{a})\underline{B}(\bar{a}) & & \underline{B}(\bar{a}) \\ \hline \dfrac{\partial\, \underline{A}_F^k(\bar{a})}{\partial \bar{a}} & \dfrac{\partial}{\partial \bar{a}}\left[\underline{A}_F^{k-1}(\bar{a})\underline{B}(\bar{a})\right] & \cdots & \dfrac{\partial\, \underline{B}(\bar{a})}{\partial \bar{a}} \end{array}\right]_{2n\times(n+k)} .$$

$$X \begin{bmatrix} \underline{x}(0) \\ v(0) \\ \cdot \\ \cdot \\ \cdot \\ v(k-1) \end{bmatrix}_{(n+k)x1} \quad (72)$$

If

$$\underline{V}_X^T(k) = \left[\underline{x}(0)^T, v(0),\ v(1), \cdots,\ v(k-1)\right]$$
$$= \left[\underline{x}^T(0),\ \underline{V}^T\right] \quad (73)$$

and

$$\underline{E}^T(k) = \left[\begin{array}{c|c|c|c} \underline{A}_F^k(\bar{a}) & \underline{A}_F^{k-1}(\bar{a})\underline{B}(\bar{a}) & & \underline{B}(\bar{a}) \\ \hline \dfrac{\partial\, \underline{A}_F^k(\bar{a})}{\partial\, \bar{a}} & \dfrac{\partial}{\partial\, \bar{a}}\left[\underline{A}_F^{k-1}(a)\underline{B}(a)\right] & \cdots & \dfrac{\partial\, \underline{B}(\bar{a})}{\partial\, \bar{a}} \end{array}\right] \quad (74)$$

then

$$\hat{\underline{X}}_A(k) = \underline{E}^T(k)\underline{V}_X(k) \quad (75)$$

and Eq (64) can be written as

$$K = \sum_{k=1}^{N} \underline{V}_X^T(k)\underline{E}(k)\underline{hC}^T\underline{R}^{-1}\underline{Ch}^T\underline{E}^T(k)\underline{V}_X(k) \quad (76)$$

Let

$$\underline{W}(k) = \underline{E}(k)\underline{hC}^T\underline{R}^{-1}\underline{Ch}^T\underline{E}^T(k) \quad (77)$$

then

$$K = \sum_{k=1}^{N} \underline{V}_X^T(k)\underline{W}(k)\underline{V}_X(k) \quad (78)$$

It is shown later in this section that it is advantageous to

express K in a quadratic form without the summation. This is accomplished by expressing K as

$$K = \underline{V}_X^T(N)\underline{W}_N\underline{V}_X(N) \tag{79}$$

where

$$\underline{V}_X^T(N) = \left[\underline{x}^T(0),\ v(0),\ v(1), \cdots,\ v(N-1)\right] \tag{80}$$

and

$$\underline{W}_N = \left[\begin{array}{c|c} \underline{W}(1)_{(n+1)x(n+1)} & \underline{0} \\ \hline \underline{0} & \underline{0}_{(N-1)x(N-1)} \end{array}\right]_{(n+N)x(n+N)}$$

$$+ \left[\begin{array}{c|c} \underline{W}(2)_{(n+2)x(n+2)} & \underline{0} \\ \hline \underline{0} & \underline{0}_{(N-2)x(N-2)} \end{array}\right]_{(n+N)x(n+N)}$$

$$+ \cdots$$

$$+ \left[\underline{W}(N)\right]_{(n+N)x(n+N)} \tag{81}$$

Substituting the values for $\underline{W}(k)$ into Eq (81) yields Eq (82) which is shown on the next page. If $\underline{x}(0) = \underline{0}$, the first n columns and n rows of $\underline{W}_N$ are zero and can be deleted. Let the remaining matrix be $\underline{W}_N'$ and the elements of $\underline{W}_N'$ are

$$w_{ij}' = \sum_{k=\max(i,j)}^{N} \frac{\partial}{\partial \bar{a}}\left[\underline{A}_F^{k-i}(\bar{a})\underline{B}(\bar{a})\right]^T \underline{C}^T \underline{R}^{-1} \underline{C} \cdot \frac{\partial}{\partial \bar{a}}\left[\underline{A}_F^{k-j}(\bar{a})\underline{B}(\bar{a})\right] \tag{83}$$

$$
\underline{W}_N = \left[\begin{array}{c|c|c}
\sum_{k=1}^{N} \frac{\partial\left[\underline{A}_F^k(\bar{a})\right]^T}{\partial \bar{a}} \underline{C}^T \underline{R}^{-1} \underline{C} \frac{\partial \underline{A}_F^k(\bar{a})}{\partial \bar{a}} & \sum_{k=1}^{N} \frac{\partial\left[\underline{A}_F^k(\bar{a})\right]^T}{\partial \bar{a}} \underline{C}^T \underline{R}^{-1} \underline{C} \frac{\partial}{\partial \bar{a}}\left[\underline{A}_F^{k-1}(\bar{a})\underline{B}(\bar{a})\right] & \cdots \\
\hline
\sum_{k=1}^{N} \frac{\partial}{\partial \bar{a}}\left[\underline{A}_F^{k-1}(\bar{a})\underline{B}(\bar{a})\right]^T \underline{C}^T \underline{R}^{-1} \underline{C} \frac{\partial \underline{A}_F^k(\bar{a})}{\partial \bar{a}} & & \\
\hline
\sum_{k=2}^{N} \frac{\partial}{\partial \bar{a}}\left[\underline{A}_F^{k-2}(\bar{a})\underline{B}(\bar{a})\right]^T \underline{C}^T \underline{R}^{-1} \underline{C} \frac{\partial \underline{A}_F^k(\bar{a})}{\partial \bar{a}} & & \\
\hline
\vdots & & \\
\hline
\sum_{k=N}^{N} \frac{\partial}{\partial \bar{a}}\left[\underline{A}_F^{k-N}(\bar{a})\underline{B}(\bar{a})\right]^T \underline{C}^T \underline{R}^{-1} \underline{C} \frac{\partial \underline{A}_F^k(\bar{a})}{\partial \bar{a}} & \cdots & \cdots
\end{array}\right]
$$

$$
\left.
\begin{array}{c|c}
\cdots & \displaystyle\sum_{k=N}^{N} \frac{\partial}{\partial \bar{a}}\left[\underline{A}_F^{k}(\bar{a})\right]^T \underline{C}^T \underline{R}^{-1} \underline{C} \frac{\partial}{\partial \bar{a}}\left[\underline{A}_F^{k-N}(\bar{a})\underline{B}(\bar{a})\right] \\
\hline
 & \vdots \\
\hline
\cdots & \displaystyle\sum_{k=N}^{N} \frac{\partial}{\partial \bar{a}}\left[\underline{A}_F^{k-N}(\bar{a})\underline{B}(\bar{a})\right]^T \underline{C}^T \underline{R}^{-1} \underline{C} \frac{\partial}{\partial \bar{a}}\left[\underline{A}_F^{k-N}(\bar{a})\underline{B}(\bar{a})\right]
\end{array}
\right] \tag{82}
$$

It is discussed later how to handle the case in which the optimal initial state is desired rather than $\underline{x}(0) = \underline{0}$.

The information scalar is then given by

$$M = \underline{V}^T \underline{W}_N' \underline{V} + J(\underline{F}_y) \tag{84}$$

where $J(\underline{F}_y)$ is given by Eq (61). For the scalar input case, this equation can be further simplified.

$$\begin{aligned} &\mathrm{tr}\left[\underline{h}^T \underline{A}_A^{N-j} \underline{B}_A \underline{B}_A^T (\underline{A}_A^T)^{N-j} \underline{hC}^T \underline{R}^{-1} \underline{C}\right] \\ &\qquad = \mathrm{tr}\left[\left[\underline{h}^T \underline{A}_A^{N-j} \underline{B}_A\right]\left[\underline{B}_A^T (\underline{A}_A^T)^{N-j} \underline{hC}^T \underline{R}^{-1} \underline{C}\right]\right] \\ &\qquad = \underline{B}_A^T (\underline{A}_A^T)^{N-j} \underline{hC}^T \underline{R}^{-1} \underline{Ch}^T \underline{A}_A^{N-j} \underline{B}_A \end{aligned} \tag{85}$$

because for $\underline{X}$, $\underline{Y}$ both n-dimensional vectors, $\mathrm{tr}\underline{XY}^T = \underline{Y}^T\underline{X}$; therefore,

$$J(\underline{F}_y) = \sum_{j=1}^{N} j\underline{F}_y \underline{RF}_y{}^T \underline{B}_A^T (\underline{A}_A^T)^{N-j} \underline{hC}^T \underline{R}^{-1} \underline{Ch}^T \underline{A}_A^{N-j} \underline{B}_A \tag{86}$$

To maximize M a straight gradient approach could be used, but as mentioned earlier, this would be very difficult. A simplified method can be used because of the special structure of Eq (84) and is discussed in the next subsection.

Maximum Value of M. It is general knowledge that, given a quadratic $\underline{X}^T\underline{AX}$ where $\underline{X}$ is an n-dimensional vector, $\underline{A}$ is an nxn matrix, and $\underline{X}^T\underline{X} = 1$, the maximum value of $\underline{X}^T\underline{AX}$ is λ_m, the maximum eigenvalue of $\underline{A}$ (Ref 22). The value of $\underline{X}$

that gives the maximum of $\underline{X}^T\underline{AX}$ is the unit eigenvector corresponding to λ_m. It is this fact that is used to simplify computing the maximum of M.

From the work leading up to Eq (84) it is shown that $\underline{W}_N'$ and $J(\underline{F}_y)$ are functions of $\underline{F}_y$ only and not of the input controls while $\underline{V}$ is naturally a function of only the input controls. This implies that for a given value of $\underline{F}_y$, the matrix $\underline{W}_N'$ and scalar $J(\underline{F}_y)$ are determined. Using the presented theory, if $\underline{V}^T\underline{V} = 1$ the maximum value of M would be

$$M_m(\underline{F}_y) = \lambda_{max}(\underline{F}_y) + J(\underline{F}_y) \tag{87}$$

where $M_m(\underline{F}_y)$ is the maximum value of M for a given $\underline{F}_y$, $\lambda_{max}(\underline{F}_y)$ is the maximum eigenvalue of $\underline{W}_N'$ for a given $\underline{F}_y$.

For the problem considered in this research, the constraint on the controls is given in Eq (18) which is

$$\sum_{k=0}^{N-1} v^2(k) \leq E \tag{18}$$

This can be expressed as

$$\underline{V}^T\underline{V} \leq E \tag{88}$$

however, in order to use the above approach the constraint on the external controls would have to be

$$\underline{V}^T\underline{V} = E \tag{89}$$

It is shown in Appendix D that for the cost function in-

volved, Eq (84), all the energy will be used and in fact $\underline{V}^T\underline{V} = E$ is valid.

Since $\underline{V}^T\underline{V}$ equals E instead of one,

$$\frac{\underline{V}^T}{\sqrt{E}} \frac{\underline{V}}{\sqrt{E}} = 1 \tag{90}$$

and

$$\max\left[\frac{\underline{V}^T}{\sqrt{E}} \underline{W}_N' \frac{\underline{V}}{\sqrt{E}}\right] = \lambda_{max}(\underline{F}_y) \tag{91}$$

then

$$\max \underline{V}^T\underline{W}_N'\underline{V} = \lambda_{max}(\underline{F}_y)E \tag{92}$$

Therefore, the maximum value of M for a given $\underline{F}_y$ and energy constraint E is

$$M_m(\underline{F}_y) = \lambda_{max}(\underline{F}_y)E + J(\underline{F}_y) \tag{93}$$

The value of the control sequence $\underline{V}$ that gives this maximum value is the eigenvector of magnitude $\sqrt{E}$ of $\underline{W}_N'$ corresponding to the eigenvalue $\lambda_{max}(\underline{F}_y)$.

The above work shows that if an optimal feedback matrix had been determined, finding the optimal control sequence is relatively easy. The difficult step is to determine the optimal $\underline{F}_y$ that maximizes $M_m(\underline{F}_y)$ while constraining the eigenvalues or poles of the feedback system to a given constraint space. The overall optimal information scalar will be called M_{opt}, that is

$$M_{opt} = \max_{\underline{F}_y} M_m(\underline{F}_y) \tag{94}$$

under the constraint that the maximum energy is E and the eigenvalues remain in Φ.

The procedure for calculating the optimal values of $\underline{F}_y$ and $\underline{V}$ can be restated for clarity. Use an optimization technique, which will be shown in the next section, to compute the optimal $\underline{F}_y$ and it will be called $\underline{F}_{yopt}$. Once $\underline{F}_{yopt}$ is computed, calculate $\underline{W}_N'$ and its maximum eigenvalue, $\lambda_{max}(\underline{F}_{yopt})$. Next compute the eigenvector of $\underline{W}_N'$ corresponding to $\lambda_{max}(\underline{F}_{yopt})$, which is called $\underline{V}_{opt}$, and scale it so that $\underline{V}_{opt}^T\underline{V}_{opt} = E$. The vector $\underline{V}_{opt}$ is the optimal value for the control sequence. As mentioned, the difficult part is finding $\underline{F}_{yopt}$ and the next section discusses the approach taken to obtain this 1xr matrix.

For the case where $\underline{x}(0) \neq \underline{0}$, the optimal initial state vector is computed along with the optimal control sequence. A constraint must be placed on the initial condition vector as well as the controls. The information scalar for this case is

$$M = \underline{V}_X^T\underline{W}_N\underline{V}_X + J(\underline{F}_y) \tag{95}$$

and assume the combined constraint on $\underline{V}_X$ is $\underline{V}_X^T\underline{V}_X \leq E'$, then

$$M_m(\underline{F}_y) = \lambda_{max}(\underline{F}_y)E' + J(\underline{F}_y) \tag{96}$$

where $\lambda_{max}(\underline{F}_y)$ is now the maximum eigenvalue of $\underline{W}_N$ and $\underline{V}_X$ is the corresponding eigenvector. The procedure for finding

the $\underline{F}_y$ to maximize $\lambda_{max}(\underline{F}_y)$ that will be developed directly applies to this situation. The optimal $\underline{V}_X$ then contains both the external control sequence, $\underline{V}$, and initial state vector, $\underline{x}(0)$.

It is also possible to put assigned weighting values on the elements of the initial state vector and control sequence. Let

$$E' = h_1^2 x_1^2(0) + h_2^2 x_2^2(0) + \cdots + h_{n+1}^2 v^2(1) + h_{n+2}^2 v^2(2) + \cdots + h_{n+N-1}^2 v^2(N-1) \quad (97)$$

Then if

$$\underline{\underline{H}} = \begin{bmatrix} h_1 & & & & \\ & h_2 & & \underline{0} & \\ & & \cdot & & \\ & & & \cdot & \\ & \underline{0} & & & h_{n+N-1} \end{bmatrix} \quad (98)$$

$$E' = (\underline{\underline{H}}\underline{V}_X)^T(\underline{\underline{H}}\underline{V}_X) \quad (99)$$

For this case

$$M = (\underline{\underline{H}}\underline{V}_X)^T \underline{W}_N (\underline{\underline{H}}\underline{V}_X) + J(\underline{F}_y)$$
$$= \underline{V}_X^T \left[\underline{\underline{H}}^T \underline{W}_N \underline{\underline{H}}\right] \underline{V}_X + J(\underline{F}_y) \quad (100)$$

and $\lambda_{max}(\underline{F}_y)$ is now the maximum eigenvalue of $\underline{\underline{H}}^T \underline{W}_N \underline{\underline{H}}$ and $\underline{V}_X$ is the corresponding eigenvalue.

Gradient Technique For Computing $\underline{F}_{yopt}$

For a given $\underline{F}_y$, the maximum value of M is

$$M_m(\underline{F}_y) = \lambda_{max}(\underline{F}_y)E + J(\underline{F}_y) \tag{93}$$

A first order gradient technique is used to find the $\underline{F}_y$ that maximizes $M_m(\underline{F}_y)$. This requires the computation of the partial

$$\left.\frac{\partial M_m(\underline{F}_y)}{\partial \underline{F}_y^T}\right|_{\underline{F}_y^T = \underline{F}_{yc}^T}$$

which from Eq (93) can be computed as follows:

$$\left.\frac{\partial M_m(\underline{F}_y)}{\partial \underline{F}_y^T}\right|_{\underline{F}_y^T = \underline{F}_{yc}^T} = E\left.\frac{\partial \lambda_{max}(\underline{F}_y)}{\partial \underline{F}_y^T}\right|_{\underline{F}_y^T = \underline{F}_{yc}^T} + \left.\frac{\partial J(\underline{F}_y)}{\partial \underline{F}_y^T}\right|_{\underline{F}_y^T = \underline{F}_{yc}^T} \tag{101}$$

where $\underline{F}_{yc}$ is the current value of $\underline{F}_y$. It can now be seen that, as mentioned earlier, the partial of the information scalar with respect to only the feedback 1xr matrix is required. This significantly reduces the computation required since the dimension of $\underline{F}_y$ is much smaller than the dimension of $\underline{V}$.

The partials $\frac{\partial (\cdot)}{\partial \underline{F}_y^T}$ are evaluated at the current iteration value of $\underline{F}_y$ so the notation indicating the point of evaluation is henceforth eliminated. The partial of $\lambda_{max}(\underline{F}_y)$ with respect to $\underline{F}_y$ can be obtained from

$$\frac{\partial \lambda_{max}(\underline{F}_y)}{\partial \underline{F}_y^T} = \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial \lambda_{max}(\underline{F}_y)}{\partial w'_{ij}} \frac{\partial w'_{ij}}{\partial \underline{F}_y^T} \tag{102}$$

where w'_{ij} are the elements of $\underline{W}'_N$. Symmetry can be used to simplify the calculations and Eq (102) becomes:

$$\frac{\partial \lambda_{max}(\underline{F}_y)}{\partial \underline{F}_y^T} = \begin{cases} 2 \sum_{i=1}^{N} \sum_{j=1}^{i-1} \frac{\partial \lambda_{max}(\underline{F}_y)}{\partial w'_{ij}} \frac{\partial w'_{ij}}{\partial \underline{F}_y^T} \\ + \sum_{i=1}^{N} \frac{\partial \lambda_{max}(\underline{F}_y)}{\partial w'_{ij}} \frac{\partial w'_{ij}}{\partial \underline{F}_y^T} \end{cases} \tag{103}$$

To obtain $\frac{\partial \lambda_{max}(\underline{F}_y)}{\partial w'_{ij}}$, the fact that $\underline{W}'_N$ is real and symmetric is used. This means that there can always be found an orthogonal transformation matrix $\underline{\Upsilon}$ such that

$$\underline{\Lambda}_N = \underline{\Upsilon}^T \underline{W}'_N \underline{\Upsilon} \tag{104}$$

where $\underline{\Lambda}_N$ is a diagonal NxN matrix with components being the eigenvalues, λ's, of $\underline{W}'_N$, $\underline{\Upsilon}$ is an NxN orthogonal matrix such that $\underline{\Upsilon}^T = \underline{\Upsilon}^{-1}$ and whose columns are the unit eigenvectors, $\underline{e}$'s, of $\underline{W}'_N$. Choosing λ_1 to be λ_{max} gives

$$\begin{bmatrix} \lambda_{max} & & & \underline{0} \\ & \lambda_2 & & \\ & & \ddots & \\ \underline{0} & & & \lambda_N \end{bmatrix} = \begin{bmatrix} \underline{e}_{max}^T \\ \hline \underline{e}_2^T \\ \hline \vdots \\ \hline \underline{e}_N^T \end{bmatrix} \underline{W}'_N \left[\underline{e}_{max} \,\middle|\, \underline{e}_2 \,\middle|\, \cdots \,\middle|\, \underline{e}_N \right] \tag{105}$$

where $\underline{e}_k$ is the normalized eigenvector corresponding to the eigenvalue λ_k. It is easily seen that

$$\lambda_{max}(\underline{F}_y) = \underline{e}_{max}{}^T \underline{W}_N' \underline{e}_{max} \tag{106}$$

Moreover, Porter and Crossley (Ref 26) show that

$$\frac{\partial \lambda_{max}(\underline{F}_y)}{\partial w_{ij}'} = e_{max_i} e_{max_j} \tag{107}$$

where e_{max_k} is the k-th element of $\underline{e}_{max}$. This means that the eigenvector corresponding to the maximum eigenvalue of $\underline{W}_N'$ must be calculated to obtain the gradient.

In order to calculate $\underline{e}_{max}$, $\underline{W}_N'$ must be computed first. From Eq (83) it can be seen that $\underline{W}_N'$ is symmetric and real. Also from Eq (64) it can be seen that $K \geq 0$, so $\underline{W}_N'$ is positive semidefinite.

To calculate the $\frac{n(n+1)}{2}$ independent terms in $\underline{W}_N'$, the partials, $\frac{\partial}{\partial \bar{a}}\left[\underline{A}_F^{k-i}(\bar{a})\underline{B}(\bar{a})\right]$, have to be evaluated and $\underline{A}_F(\bar{a})$ is given by Eq (66). This computation can be done iteratively.

$$\begin{aligned}\frac{\partial}{\partial \bar{a}}\left[\underline{A}_F^m(\bar{a})\underline{B}(\bar{a})\right] &= \frac{\partial}{\partial \bar{a}}\left[\underline{A}_F(\bar{a})\underline{A}_F^{m-1}(\bar{a})\underline{B}(\bar{a})\right] \\ &= \underline{A}_F(\bar{a})\frac{\partial}{\partial \bar{a}}\left[\underline{A}_F^{m-1}(\bar{a})\underline{B}(\bar{a})\right] \\ &\quad + \frac{\partial \underline{A}_F(\bar{a})}{\partial \bar{a}}\left[\underline{A}_F^{m-1}(\bar{a})\underline{B}(\bar{a})\right]\end{aligned} \tag{108}$$

where

$$\underline{A}_F^O(\bar{a})\underline{B}(\bar{a}) = \underline{B}(\bar{a}) \tag{109}$$

and

$$\frac{\partial \underline{A}_F(\bar{a})}{\partial \bar{a}} = \frac{\partial\, \underline{A}(\bar{a})}{\partial\, \bar{a}} + \frac{\partial \underline{B}(\bar{a})}{\partial\, \bar{a}}\underline{F}_y\underline{C} \tag{110}$$

Also

$$\begin{aligned} w'_{ij} &= \sum_{k=\max(i,j)}^{N} \frac{\partial}{\partial\, \bar{a}}\left[\underline{A}_F^{k-i}(\bar{a})\underline{B}(\bar{a})\right]^T \underline{C}^T\underline{R}^{-1}\underline{C}\frac{\partial}{\partial \bar{a}}\left[\underline{A}_F^{k-j}(\bar{a})\underline{B}(\bar{a})\right] \\ &= \sum_{k=\max(i+1,j+1)}^{N} \frac{\partial}{\partial\, \bar{a}}\left[\underline{A}_F^{k-i}(\bar{a})\underline{B}(\bar{a})\right]^T \underline{C}^T\underline{R}^{-1}\underline{C}\frac{\partial}{\partial\, \bar{a}}\left[\underline{A}_F^{k-j}(\bar{a})\underline{B}(\bar{a})\right] \\ &\quad + \frac{\partial}{\partial\, \bar{a}}\left[\underline{A}_F^{k-i}(\bar{a})\underline{B}(\bar{a})\right]^T \underline{C}^T\underline{R}^{-1}\underline{C}\frac{\partial}{\partial\, \bar{a}}\left[\underline{A}_F^{k-j}(\bar{a})\underline{B}(\bar{a})\right]_{k=\max(i,j)} \\ &= w'_{i+1,j+1} \\ &\quad + \frac{\partial}{\partial\, \bar{a}}\left[\underline{A}_F^{k-i}(\bar{a})\underline{B}(\bar{a})\right]^T \underline{C}^T\underline{R}^{-1}\underline{C}\frac{\partial}{\partial \bar{a}}\left[\underline{A}_F^{k-j}(\bar{a})\underline{B}(\bar{a})\right]_{k=\max(i,j)} \end{aligned} \tag{111}$$

Therefore, first calculate the elements in the bottom row of $\underline{W}'_N$ and then use Eq (111) to obtain the other elements in the lower triangle of $\underline{W}'_N$. Knowing the elements of $\underline{W}'_N$ it is then possible to get $\lambda_{max}(\underline{F}_y)$ and $\underline{e}_{max}$, the unit length eigenvector of $\underline{W}'_N$, by means of existing algorithms referenced in the book by K. Rektorys (Ref 29).

The $\dfrac{\partial\, w'_{ij}}{\partial\, \underline{F}_y^T}$ required in Eq (102) must now be obtained.

Using Eq (83)

$$\frac{\partial w'_{Nj}}{\partial \underline{F}_y^T} = \frac{\partial \underline{B}^T(\bar{a})}{\partial \bar{a}} \underline{C}^T \underline{R}^{-1} \underline{C} \frac{\partial}{\partial \underline{F}_y^T} \left[\frac{\partial}{\partial \bar{a}} \left[\underline{A}_F^{N-j}(\bar{a}) \underline{B}(\bar{a}) \right] \right] \qquad (112)$$

and from Eq (111) the remaining terms can be obtained iteratively by means of

$$\frac{\partial w'_{ij}}{\partial \underline{F}_y^T} = \frac{\partial w'_{i+1,j+1}}{\partial \underline{F}_y^T} \qquad (113)$$

$$+ \frac{\partial}{\partial \bar{a}} \left[\underline{A}_F^{k-i}(\bar{a}) \underline{B}(\bar{a}) \right]^T \underline{C}^T \underline{R}^{-1} \underline{C} \frac{\partial}{\partial \underline{F}_y^T} \left[\frac{\partial}{\partial \bar{a}} \left[\underline{A}_F^{k-j}(\bar{a}) \underline{B}(\bar{a}) \right] \right]_{k=\max(i,j)}$$

$$+ \frac{\partial}{\partial \bar{a}} \left[\underline{A}_F^{k-j}(\bar{a}) \underline{B}(\bar{a}) \right]^T \underline{C}^T \underline{R}^{-1} \underline{C} \frac{\partial}{\partial \underline{F}_y^T} \left[\frac{\partial}{\partial \bar{a}} \left[\underline{A}_F^{k-i}(\bar{a}) \underline{B}(\bar{a}) \right] \right]_{k=\max(i,j)}$$

where

$\frac{\partial}{\partial \underline{F}_y^T} \left[\frac{\partial}{\partial \bar{a}} \left[\underline{A}_F^{l-m}(\bar{a}) \underline{B}(\bar{a}) \right] \right]$ is an nxr matrix and has to be derived. From Eq (108)

$$\frac{\partial}{\partial \underline{F}_y^T} \left[\frac{\partial}{\partial \bar{a}} \left[\underline{A}_F^{m}(\bar{a}) \underline{B}(\bar{a}) \right] \right] = \underline{A}_F(\bar{a}) \frac{\partial}{\partial \underline{F}_y^T} \left[\frac{\partial}{\partial \bar{a}} \left[\underline{A}_F^{m-1}(\bar{a}) \underline{B}(\bar{a}) \right] \right]$$

$$+ \frac{\partial}{\partial \underline{F}_y^T} \underline{A}_F(\bar{a}) \frac{\partial}{\partial \bar{a}} \left[\underline{A}_F^{m-1}(\bar{a}) \underline{B}(\bar{a}) \right]$$

$$+ \frac{\partial}{\partial \underline{F}_y^T} \left[\frac{\partial \underline{A}_F(\bar{a})}{\partial \bar{a}} \right] \left[\underline{A}_F^{m-1}(\bar{a}) \underline{B}(\bar{a}) \right]$$

$$+ \frac{\partial \underline{A}_F(\bar{a})}{\partial \bar{a}} \frac{\partial}{\partial \underline{F}_y{}^T}\left[\underline{A}_F^{m-1}(\bar{a})\underline{B}(\bar{a})\right] \qquad (114)$$

where

$$\frac{\partial}{\partial \underline{F}_y{}^T}\left[\frac{\partial}{\partial \bar{a}}\left[\underline{A}_F^0(\bar{a})\underline{B}(\bar{a})\right]\right] = \frac{\partial}{\partial \underline{F}_y{}^T} \frac{\partial \underline{B}(a)}{\partial \bar{a}} = \underline{0} \qquad (115)$$

Using the proofs in Appendix E, the following equations can readily be generated.

$$\frac{\partial}{\partial \underline{F}_y{}^T}\underline{A}_F(\bar{a})\frac{\partial}{\partial \bar{a}}\left[\underline{A}_F^{m-1}(\bar{a})\underline{B}(\bar{a})\right]$$

$$= \underline{B}(\bar{a})\frac{\partial}{\partial \bar{a}}\left[\underline{A}_F^{m-1}(\bar{a})\underline{B}(\bar{a})\right]^T\underline{C}^T \qquad (116)$$

$$\frac{\partial}{\partial \underline{F}_y{}^T}\left[\frac{\partial \underline{A}_F(\bar{a})}{\partial \bar{a}}\right]\left[\underline{A}_F^{m-1}(\bar{a})\underline{B}(\bar{a})\right]$$

$$= \frac{\partial \underline{B}(\bar{a})}{\partial \bar{a}}\left[\underline{A}_F^{m-1}(\bar{a})\underline{B}(\bar{a})\right]^T\underline{C}^T \qquad (117)$$

$$\frac{\partial \underline{A}_F(\bar{a})}{\partial \bar{a}} \frac{\partial}{\partial \underline{F}_y{}^T}\left[\underline{A}_F^{m-1}(\bar{a})\underline{B}(\bar{a})\right]$$

$$= \frac{\partial \underline{A}_F(\bar{a})}{\partial \bar{a}} \sum_{l=1}^{m-1} \underline{A}_F^{l-1}(\bar{a})\underline{B}(\bar{a})\underline{B}^T(\bar{a})\left[\underline{A}_F^T(\bar{a})\right]^{m-1-l}\underline{C}^T \qquad (118)$$

Equations (116) through (118) and Eq (108) can now be substituted into Eq (114). This seems to be complex, but the equations are iterative and can be calculated rapidly on a computer.

All the tools are now available to calculate $\dfrac{\partial \lambda_{max}(\underline{F}_y)}{\partial \underline{F}_y^T}$

since

$$\frac{\partial \lambda_{max}(\underline{F}_y)}{\partial \underline{F}_y^T} = \left\{ \begin{array}{l} 2\displaystyle\sum_{i=1}^{N} \sum_{j=1}^{i-1} \frac{\partial \lambda_{max}(\underline{F}_y)}{\partial w'_{ij}} \frac{\partial w'_{ij}}{\partial \underline{F}_y^T} \\ + \displaystyle\sum_{i=1}^{N} \frac{\partial \lambda_{max}(\underline{F}_y)}{\partial w'_{ij}} \frac{\partial w'_{ij}}{\partial \underline{F}_y^T} \end{array} \right. \tag{103}$$

and

$\dfrac{\partial \lambda_{max}(\underline{F}_y)}{\partial w'_{ij}}$ is evaluated via Eq (107),

$\dfrac{\partial w'_{ij}}{\partial \underline{F}_y^T}$ by means of Eqs (112) and (113),

$\dfrac{\partial}{\partial \underline{F}_y^T}\left[\dfrac{\partial}{\partial \bar{a}}\left[\underline{A}_F^m(\bar{a})\underline{B}(\bar{a})\right]\right]$ by Eqs (108), (116) through (118)

and (114),

and

$\dfrac{\partial}{\partial \bar{a}}\left[\underline{A}_F(\bar{a})\underline{B}(\bar{a})\right]$ through Eqs (108) to (110).

The next step is to calculate $\dfrac{\partial J(\underline{F}_y)}{\partial \underline{F}_y^T}$. Rewriting Eq (86) yields

$$J(\underline{F}_y) = \sum_{j=1}^{N} j\underline{F}_y \underline{RF}_y{}^T \underline{B}_A{}^T (\underline{A}_A^T)^{N-j} \underline{hC}^T \underline{R}^{-1} \underline{Ch}^T \underline{A}_A^{N-j} \underline{B}_A \tag{86}$$

so

$$\frac{\partial J(\underline{F}_y)}{\partial \underline{F}_y^T} = \sum_{j=1}^{N} \Big[2j\underline{F}_y \underline{R}\underline{B}_A^T(\underline{A}_A^T)^{N-j}\underline{h}\underline{C}^T\underline{R}^{-1}\underline{C}\underline{h}^T\underline{A}_A^{N-j}\underline{B}_A$$

$$+ 2j\underline{F}_y\underline{R}\underline{F}_y^T\underline{B}_A^T(\underline{A}_A^T)^{N-j}\underline{h}\underline{C}^T\underline{R}^{-1}\underline{C}\frac{\partial}{\partial \underline{F}_y^T}\Big[\underline{h}^T\underline{A}_A^{N-j}\underline{B}_A\Big]\Big]$$

$$= 2\sum_{j=1}^{N} j\underline{F}_y\underline{R}\Big[\underline{B}_A^T(\underline{A}_A^T)^{N-j}\underline{h}\underline{C}^T\underline{R}^{-1}\underline{C}\underline{h}^T\underline{A}_A^{N-j}\underline{B}_A$$

$$+ \underline{F}_y^T\underline{B}_A^T(\underline{A}_A^T)^{N-j}\underline{h}\underline{C}^T\underline{R}^{-1}\underline{C}\frac{\partial}{\partial \underline{F}_y^T}\Big[\underline{h}^T\underline{A}_A^{N-j}\underline{B}_A\Big]\Big] \qquad (119)$$

From Eq (69) and the definition of $\underline{h}$, Eq (119) becomes:

$$\frac{\partial J(\underline{F}_y)}{\partial \underline{F}_y^T} = 2\sum_{j=1}^{N} j\underline{F}_y\underline{R}\Bigg(\frac{\partial}{\partial \bar{a}}\Big[\underline{A}_F^{N-j}(\bar{a})\underline{B}(\bar{a})\Big]^T\underline{C}^T\underline{R}^{-1}\underline{C}$$

$$\cdot \frac{\partial}{\partial \bar{a}}\Big[\underline{A}_F^{N-j}(\bar{a})\underline{B}(\bar{a})\Big] + \underline{F}_y^T\frac{\partial}{\partial \bar{a}}\Big[\underline{A}_F^{N-j}(\bar{a})\underline{B}(\bar{a})\Big]^T$$

$$\cdot \underline{C}^T\underline{R}^{-1}\underline{C}\frac{\partial}{\partial \underline{F}_y^T}\Big[\frac{\partial}{\partial \bar{a}}\Big[\underline{A}_F^{N-j}(\bar{a})\underline{B}(\bar{a})\Big]\Big]\Bigg) \qquad (120)$$

Equations (108),(109),(116) through (118), and (114) are used with Eq (120) to evaluate $\frac{\partial J(\underline{F}_y)}{\partial \underline{F}_y^T}$ for a given $\underline{F}_y$. It can easily be seen from Eq (120) that with no feedback,

$$\frac{\partial J(\underline{0})}{\partial \underline{F}_y^T} = \underline{0} \ .$$

All the information is available to compute $\frac{\partial M_m(\underline{F}_y)}{\partial \underline{F}_y^T}$.

The following equation

$$M = \overline{M}_o + \left.\frac{\partial M_m(\underline{F}_y)}{\partial \underline{F}_y^T}\right|_{\underline{F}_y = \underline{F}_{yc}} \Delta\underline{F}_y + \text{H.O.T.} \tag{121}$$

where H.O.T. are the higher order terms of the Taylor expansion, $\Delta\underline{F}_y$ is the step size in the feedback space, and $\overline{M}_o$ is the cost function before the step was taken, shows that the step should be in the same direction as the gradient, if there are no constraints, inorder to maximize M. There are in fact constraints so the standard gradient approach to maximizing M must be modified.

Transfer Gradient into Complex Space. The constraints are defined in the complex space of eigenvalues. An example of a constraint is

$$|\lambda_i| < \overline{R} \quad \text{for } 0 < \overline{R} \leq 1 \quad \text{and } i=1, 2, \cdots, n \tag{122}$$

which states that all eigenvalues must be within a circle whose maximum radius is one. This is a requirement for the system to be stable. Adding another constraint which is

$$|\text{Imag}\,\lambda| \leq \text{real}\,\lambda \tag{123}$$

$$0 \leq \text{real}\,\lambda < 1 \tag{124}$$

represents a sector as shown in Figure 3. The shaded region is Φ. This constraint represents a system which must remain stable and the frequency of oscillation of the response is restricted to be less than a desired amount.

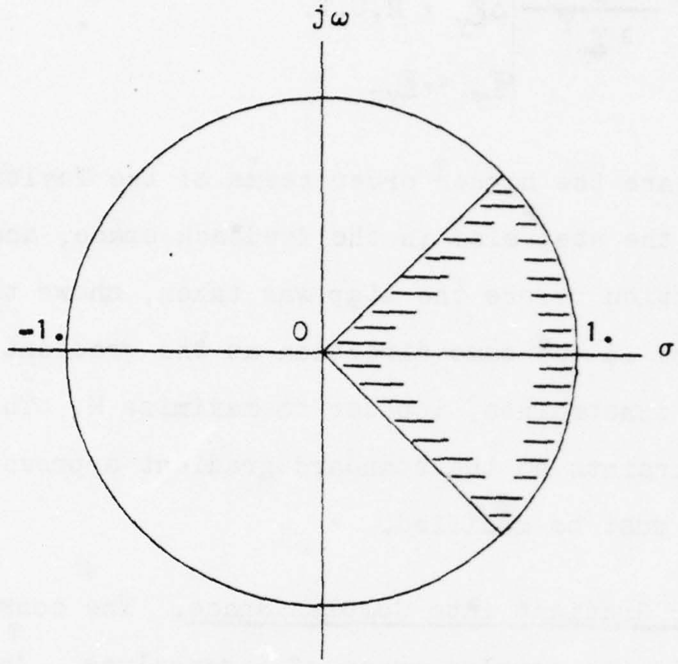


Figure 3. Constraint Space in Z-Plane; $\bar{R} = 1$

At this point the constraints are in the eigenvalue space and the gradient is in the feedback vector space. It would be difficult, and insight would be lost, if one tried to transform the constraint area in the eigenvalue space to the feedback space. Instead it is proposed to transform the gradient into the complex eigenvalue space. That is, find $\dfrac{\partial M_m(\underline{F}_y)}{\partial \underline{\lambda}_F}$ where $\underline{\lambda}_F$ is the n-dimensional vector of eigenvalues of $\underline{A}_F(\bar{a})$ and $\underline{A}_F(\bar{a})$ is given be Eq (66). Now the procedure is to move in the direction of this gradient in the complex space and be sure the eigenvalues stay within the constraint space.

With this approach a $\underline{\lambda}_F$, instead of $\underline{F}_y$, will be found that maximizes M. Once $\underline{\lambda}_F$ is computed, a transformation equation is needed to compute the $\underline{F}_y$ that is required to transform the open-loop eigenvalues of $\underline{A}(\bar{a})$ to feedback eigenvalues given by $\underline{\lambda}_F$.

To obtain an equation relating the feedback eigenvalues to $\underline{F}_y$, an approach developed by Porter (Ref 25) will be used. A detailed description of the approach is presented in Appendix F. This method allows one to obtain a closed form equation for $\underline{F}_s$, where $\underline{F}_s$ is a state feedback 1xn matrix, in terms of $\underline{\lambda}_F$. State feedback means that the output is the entire state vector. For the single input case this is not a significant achievement, however, this method also applies to the multi-input case where the state feedback is now an mxn matrix. Most available methods that solve for $\underline{F}_s$ for the multi-input case are not in a closed form solution. Some of the methods as given by Porter and Crossley (Ref 26) are minimum-gain controllers, prescribed-gain controllers, and multi-stage controllers. The dyadic form controller would give a closed form solution, but the feedback matrix would have to have the dyadic form

$$\underline{F}_s = \underline{f}\underline{z}^T \tag{127}$$

where $\underline{f}$ and $\underline{z}$ are respectively mx1 and nx1 vectors. Forcing the feedback matrix to be of rank one would restrict the degrees of freedom allowed to select an optimal matrix.

The approach that is being used provides the relationship between a state feedback matrix, $\underline{F}_s$ and the output eigenvalues and not the relationship between an output feedback matrix, $\underline{F}_y$ and the eigenvalues. Since this research is concerned with $\underline{F}_y$, another equation is needed to obtain $\underline{F}_y$ from $\underline{F}_s$. It is known that

$$\underline{F}_s = \underline{F}_y\underline{C} \tag{128}$$

where $\underline{C}$ is the rxn output matrix. If $\underline{C}$ were an nxn matrix of rank n, then $\underline{F}_y$ would be

$$\underline{F}_y = \underline{F}_s\underline{C}^{-1} \tag{129}$$

However, in general this is not the case, and for this research $\underline{C}^{-1}$ does not exist. It is shown later that the pseudoinverse of $\underline{C}$ is used and that additional constraints of the eigenvalues are required to obtain $\underline{F}_y$ from $\underline{F}_s$.

Computing $\underline{F}_s$ from $\underline{\lambda}_F$ will be developed for the multi-input system. Even though this Chapter deals only with the single input case, it will save having to repeat much of the procedure for the multi-input case discussed in Chapter IV. Wherever a significant change occurs for the single input case, it will be mentioned at that time.

<u>Computing the State Feedback Equation</u>. The method of computing the state feedback equation is to transform the original state equation

$$\underline{x}(k+1) = \underline{A}(\bar{a})\underline{x}(k) + \underline{B}(\bar{a})\underline{u}(k) \tag{130}$$

to the generalized control canonical form

$$\underline{z}(k+1) = \underline{F}_c(\bar{a})\underline{z}(k) + \underline{G}_c(\bar{a})\underline{w}_c \tag{131}$$

where $\underline{F}_c(\bar{a})$ is block diagonal $(\underline{F}_{k_1}, \underline{F}_{k_2}, \cdots, \underline{F}_{k_m})$ and $\underline{G}_c(\bar{a})$ is block diagonal $(\underline{g}_{k_1}, \underline{g}_{k_2}, \cdots, \underline{g}_{k_m})$. For the reader who is unfamiliar with the generalized control canonical form it would be helpful to read Appendix F at this time. The Appendix explains the procedure described here and also gives the reasoning behind each step. Eq (131) is obtained from Eq (130) by means of three transformations:

(1) $\underline{x}(k) = \underline{Tz}(k)$ where $\underline{T}$ is an nxn matrix and det $\underline{T}$ does not equal zero. Equation (130) becomes:

$$\underline{z}(k+1) = \underline{T}^{-1}\underline{A}(\bar{a})\underline{Tz}(k) + \underline{T}^{-1}\underline{B}(\bar{a})\underline{u}(k) \tag{132}$$

(2) $\underline{u}(k) = \underline{Zv}_c(k)$ where $\underline{Z}$ is an mxm matrix and det $\underline{Z}$ does not equal zero. Equation (132) becomes:

$$\underline{z}(k+1) = \underline{T}^{-1}\underline{A}(\bar{a})\underline{Tz}(k) + \underline{T}^{-1}\underline{B}(\bar{a})\underline{Zv}_c(k) \tag{133}$$

(3) The introduction of feedback $\underline{v}_c(k) = \underline{w}_c(k) - \underline{Lz}(k)$ where $\underline{w}_c(k)$ is an mx1 external control vector and $\underline{L}$ is an mxn feedback matrix. The purpose of the $\underline{L}$ matrix is to add the necessary freedom to change the transformed $\underline{A}(\bar{a})$ matrix to the generalized control canonical form $\underline{F}_c(\bar{a})$. The $\underline{w}_c(k)$ control vector still allows one to apply a state feedback to the generalized form and that will be demonstrated later. Equation

(133) becomes:

$$\underline{z}(k+1) = (\underline{T}^{-1}\underline{A}(\bar{a})\underline{T} - \underline{T}^{-1}\underline{B}(\bar{a})\underline{ZL})\underline{z}(k) + \underline{T}^{-1}\underline{B}(\bar{a})\underline{Zw}_c(k) \quad (135)$$

$$= \underline{F}_c(\bar{a})\underline{z}(k) + \underline{G}_c(\bar{a})\underline{w}_c(k) \quad (131)$$

and

$$\underline{F}_c(\bar{a}) = \underline{T}^{-1}(\underline{A}(\bar{a})\underline{T} - \underline{B}(\bar{a})\underline{ZL}) \quad (136)$$

$$\underline{G}_c(\bar{a}) = \underline{T}^{-1}\underline{B}(\bar{a})\underline{Z} \quad (137)$$

Appendix F explains how the matrices $\underline{T}$, $\underline{Z}$, and $\underline{L}$ are derived once the desired form has been determined. In order to move the feedback eigenvalues to the desired locations, a feedback matrix must be calculated. Let

$$\underline{w}_c(k) = \underline{Hz}(k) \quad (138)$$

where $\underline{H}$ is the mxn feedback matrix that feeds the generalized state vector into the input. This will move the eigenvalues and Eq (131) becomes:

$$\underline{z}(k+1) = (\underline{F}_c(\bar{a}) + \underline{G}_c(\bar{a})\underline{H})\underline{z}(k)$$

$$= \underline{F}_d\underline{z}(k) \quad (139)$$

where $\underline{F}_d$ is the desired feedback system matrix. It is selected to be in the block diagonal form ($\underline{F}'_{d_{k_1}}$, $\underline{F}'_{d_{k_2}}$, ... , $\underline{F}'_{d_{k_m}}$) where $\underline{F}'_{d_{k_m}}$ is a matrix block which has the same dimension as the corresponding block in $\underline{F}_c$ and it is in the

companion form. Matrix $\underline{H}$ can now be evaluated and as shown in Appendix F, its form is

$$\underline{H} = \left[\begin{array}{c|c|c} h_{11}, h_{12}, \;\; , h_{1k_1} & \underline{0} & \\ \hline \underline{0} & h_{2(k_1+1)}, \;\; , h_{2(k_1+k_2)} & \cdots \\ \hline \underline{0} & \underline{0} & \end{array}\right.$$

$$\left.\begin{array}{c|c} & \underline{0} \\ \hline \cdots & \underline{0} \\ \hline & h_{m(n-k_m)}, \cdots, h_{mn} \end{array}\right] \quad (140)$$

where $k_1 + k_2 + \cdots + k_m = n$ are the control invariants (see Appendix F). For the single input case

$$\underline{H} = \left[h_{11}, h_{12}, \cdots, h_{1n}\right] \quad (141)$$

The nonzero elements in the i-th row of Eq (140) and the only row of Eq (141) are the coefficients of the characteristic polynomial of k_i and n feedback system eigenvalues respectively. This can be expressed in a summation form of the products of these eigenvalues.

$$h_{jp} = (-1)^{l-1} \sum_{i_1=1}^{k_j-l+1} \sum_{i_2=i_1+1}^{k_j-l+2} \cdots \sum_{i_l=i_{l-1}+1}^{k_m} \lambda_{F_{i_1}} \lambda_{F_{i_2}} \cdots \lambda_{F_{i_l}} \quad (142)$$

where the maximum number of summations is equal to k_m and the number of eigenvalues in each product of the summation is equal to

$$l = \sum_{i=1}^{j} k_i - p + 1 \tag{143}$$

Equations (142) and (143) show that as one computes the elements in a given row and moves from left to right, the i_l-th summation is removed first and then the i_{l-1}-th summation. This continues until only the i_1 summation remains. For the single input case, $j = 1$ and $k_m = n$. An example will better explain this procedure.

Assume $k_1 = 3$ and the desired feedback system eigenvalues are λ_{F_1}, λ_{F_2}, and λ_{F_3}. The characteristic polynomial is

$$\begin{aligned}(\lambda - \lambda_{F_1})(\lambda - \lambda_{F_2})(\lambda - \lambda_{F_3}) \\ = \lambda^3 - (\lambda_{F_1} + \lambda_{F_2} + \lambda_{F_3})\lambda^2 \\ + (\lambda_{F_1}\lambda_{F_2} + \lambda_{F_1}\lambda_{F_3} + \lambda_{F_2}\lambda_{F_3})\lambda - \lambda_{F_1}\lambda_{F_2}\lambda_{F_3}\end{aligned} \tag{144}$$

The companion matrix form that yields this characteristic equation is

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ \lambda_{F_1}\lambda_{F_2}\lambda_{F_3} & -(\lambda_{F_1}\lambda_{F_2} + \lambda_{F_1}\lambda_{F_3} + \lambda_{F_2}\lambda_{F_3}) \end{bmatrix}$$

$$\left.\begin{matrix} 0 \\ 1 \\ (\lambda_{F_1} + \lambda_{F_2} + \lambda_{F_3}) \end{matrix}\right]$$

From Appendix F it is shown that the elements in $\underline{H}$ come from the bottom rows of the companion matrices. Therefore, the elements in the first row of $\underline{H}$ for this example are

$$h_{11} = \lambda_{F_1}\lambda_{F_2}\lambda_{F_3} \tag{145}$$

$$h_{12} = -(\lambda_{F_1}\lambda_{F_2} + \lambda_{F_1}\lambda_{F_3} + \lambda_{F_2}\lambda_{F_3}) \tag{146}$$

$$h_{13} = \lambda_{F_1} + \lambda_{F_2} + \lambda_{F_3} \tag{147}$$

$$h_{14} \text{ to } h_{1n} = 0 \tag{148}$$

Using Eq (142), for h_{11}

$$l = k_1 - 1 + 1 = 3$$

$$h_{11} = (-1)^2 \sum_{i_1=1}^{1} \sum_{i_2=i_1+1}^{2} \sum_{i_3=i_2+1}^{3} \lambda_{F_{i_1}} \lambda_{F_{i_2}} \lambda_{F_{i_3}}$$

$$= \lambda_{F_1}\lambda_{F_2}\lambda_{F_3} \tag{149}$$

for h_{12}

$$l = k_1 - 2 + 1 = 2$$

$$h_{12} = (-1)^1 \sum_{i_1=1}^{2} \sum_{i_2=i_1+1}^{3} \lambda_{F_{i_1}} \lambda_{F_{i_2}}$$

$$= -(\lambda_{F_1}\lambda_{F_2} + \lambda_{F_1}\lambda_{F_3} + \lambda_{F_2}\lambda_{F_3}) \tag{150}$$

and for h_{13}

$$l = k_1 - 3 + 1 = 1$$

$$h_{13} = (-1)^0 \sum_{i_1=1}^{3} \lambda_{F_{i_1}} = \lambda_{F_1} + \lambda_{F_2} + \lambda_{F_3} \tag{151}$$

Therefore, it was shown for this example that Eq (142) is valid. The matrix $\underline{H}$ is a function of the feedback system eigenvalues.

In order to return to the original state space, it is observed that

$$\underline{u}(k) = \underline{Z}\underline{v}_c(k)$$

and from Eqs (134) and (138)

$$\begin{aligned} \underline{u}(k) &= \underline{ZHz}(k) - \underline{ZLz}(k) \\ &= \underline{ZHT}^{-1}\underline{x}(k) - \underline{ZLT}^{-1}\underline{x}(k) \\ &= \underline{Z}(\underline{H} - \underline{L})\underline{T}^{-1}\underline{x}(k) \end{aligned} \tag{152}$$

Therefore, the feedback matrix is

$$\underline{F}_s = \underline{Z}(\underline{H} - \underline{L})\underline{T}^{-1} \tag{153}$$

This is the desired equation and a means of obtaining $\underline{F}_y$ from $\underline{F}_s$ is now addressed.

<u>Computing the Output Feedback Equation</u>. Repeating Eq (128) yields

$$\underline{F}_s = \underline{F}_y\underline{C} \tag{128}$$

and postmultiplying by $\underline{C}^T$ yields

$$\underline{F}_s\underline{C}^T = \underline{F}_y\underline{CC}^T \tag{154}$$

where $\underline{CC}^T$ is of rank r. Therefore, $(\underline{CC}^T)^{-1}$ exists and

$$\underline{F}_y = \underline{F}_s\underline{C}^T(\underline{CC}^T)^{-1} \tag{155}$$

and

$$\underline{F}_y{}^T = (\underline{CC}^T)^{-1}\underline{CF}_s{}^T \tag{156}$$

The term $\underline{C}^T(\underline{CC}^T)^{-1}$ is called the pseudoinverse of $\underline{C}$ or $\underline{C}^+$. It is not an exact inverse in that even though Eq (155) gives an $\underline{F}_y$ from $\underline{F}_s$ and $\underline{C}$, using this $\underline{F}_y$ in Eq (128) will generally not yield the original $\underline{F}_s$. In this case $\underline{CC}^+ = \underline{I}$, but $\underline{C}^+\underline{C} \neq \underline{I}$. The $\underline{F}_y$ obtained is the solution that minimizes the norm of $\underline{F}_s - \underline{F}_y\underline{C}$ and whose own norm is the smallest of all solutions.

This problem occurs because $\underline{C}$ is only of rank r; there are more equations than unknowns in Eq (128). This implies that there may not be any $\underline{F}_y$'s that will satisfy Eq (128). Equation (155) gives an approximate solution and if this solution is used, the eigenvalues of $\underline{A}_F(\bar{a})$ may not actually be $\underline{\lambda}_F$ and may even fall outside of the constraint space. To correct this, the scalar equations (128) must be forced to be consistent. Consistency means that even though there are more equations than unknowns, they are such that only one unique solution will satisfy all of them. With this condition of consistency imposed on Eq (128), the $\underline{F}_y$ calculated

from Eq (155) when placed in Eq (128) would yield the given $\underline{F}_s$. As will be seen, forcing "consistency" will impose more constraints on $\underline{\lambda}_F$.

For the single input case, Eq (128) can be written as

$$\left[f_{s_1}, f_{s_2}, \cdots, f_{s_n}\right] = \left[f_{y_1}, f_{y_2}, \cdots, f_{y_r}\right]$$

$$\text{X} \left[\underline{c}_1 \,\vdots\, \underline{c}_2 \,\vdots\, \cdot \,\vdots\, \underline{c}_n\right] \tag{157}$$

Since the rank of $\underline{C}$ is r, there are r linear independent columns of $\underline{C}$. Without loss of generality, assume the first r columns are independent and the matrix consisting of only these columns is $\underline{C}'$. That is

$$\underline{C}' = \left[c_1, c_2, \cdots, c_r\right] \tag{158}$$

and the rank of $\underline{C}'$ is r. Therefore,

$$\left[f_{s_1}, f_{s_2}, \cdots, f_{s_r}\right] = \left[f_{y_1}, f_{y_2}, \cdots, f_{y_r}\right]\underline{C}' \tag{159}$$

and

$$\left[f_{y_1}, f_{y_2}, \cdots, f_{y_r}\right] = \left[f_{s_1}, f_{s_2}, \cdots, f_{s_r}\right]\underline{C}'^{-1} \tag{160}$$

There may be an optimal set of independent columns that give "better" consistency constraints. How to select this set, if it in fact does exist, is not investigated in this research and is suggested for future research. A suggestion is to select those columns with the greatest number of zeros

in it in order to simplify the computations. The remaining n-r columns of $\underline{C}$ are linearly dependent on the columns of $\underline{C}'$ so

$$\underline{c}_i = \sum_{j=1}^{r} k_{ij}\underline{c}_j \qquad (161)$$

where $i = r+1, r+2, \cdots, n$. From Eq (157) it is easily seen that

$$\begin{aligned} f_{s_i} &= \underline{F}_y \underline{c}_i \\ &= \underline{F}_y \sum_{j=1}^{r} k_{ij}\underline{c}_j \\ &= \sum_{j=1}^{r} k_{ij}\underline{F}_y\underline{c}_j \\ &= \sum_{j=1}^{r} k_{ij} f_{s_j} \end{aligned} \qquad (162)$$

The k_{ij}'s can be computed from $\underline{C}$, and Eq (162) gives the relationship that must be maintained so that the $\underline{F}_y$ computed from Eq (155) will produce the original $\underline{F}_s$ when inserted in Eq (128). Since the elements of $\underline{F}_s$ are in terms of the feedback system eigenvalues, $\underline{\lambda}_F$, as shown in Eq (153), Eq (162) is then added constraints on the feedback system eigenvalues.

<u>Computing the Gradient</u>. To find $\frac{\partial M_m(\underline{F}_y)}{\partial \underline{\lambda}_F}$ the follow-

ing equation is used

$$\left.\frac{\partial M_m(\underline{F}_y)}{\partial \underline{\lambda}_F}\right|_{\underline{\lambda}_F = \underline{\lambda}_{Fc}} = \left.\frac{\partial M_m(\underline{F}_y)}{\partial \underline{F}_y^T}\right|_{\underline{F}_y = \underline{F}_{yc}} \left.\frac{\partial \underline{F}_y^{T*}}{\partial \underline{\lambda}_F}\right|_{\underline{\lambda}_F = \underline{\lambda}_{Fc}} \tag{163}$$

where "*" denotes complex conjugate. The conjugate is required because the inner product is being taken in the complex eigenvalue space. For the single input case that is discussed in this section, $\frac{\partial \underline{F}_y^T}{\partial \underline{\lambda}_F}$ is an rxn matrix. The multi-input control case will be discussed in Chapter IV.

From Eq (153) and the fact that $\underline{Z}$ is a scalar, Z,

$$\frac{\partial \underline{F}_s^T}{\partial \underline{\lambda}_F} = (\underline{T}^{-1})^T \frac{\partial \underline{H}^T}{\partial \underline{\lambda}_F} Z \tag{164}$$

since $\underline{H}$ is the only matrix in Eq (153) that depends on $\underline{\lambda}_F$.
Therefore,

$$\frac{\partial \underline{F}_y^T}{\partial \underline{\lambda}_F} = (\underline{C}\underline{C}^T)^{-1}\underline{C}(\underline{T}^{-1})^T \frac{\partial \underline{H}^T}{\partial \underline{\lambda}_F} Z \tag{165}$$

To find $\frac{\partial \underline{H}^T}{\partial \underline{\lambda}_F}$, the partial of each element of $\underline{H}$ with respect to each element of $\underline{\lambda}_F$ must be determined. From Eq (142)

$$\frac{\partial h_{jp}}{\partial \lambda_{F_m}} = (-1)^{l-1} \sum_{i_1=1}^{k_j-l+2} \sum_{i_2=i_1+1}^{k_j-l+3} \cdots \sum_{\substack{i_{l-1}=i_{l-2}+1 \\ i_1,\, i_2, \cdots,\, i_{l-1} \neq m}}^{k_m} \lambda_{F_{i_1}} \lambda_{F_{i_1}} \cdots \lambda_{F_{i_{l-1}}} \tag{166}$$

For the example given earlier

$$\frac{\partial h_{11}}{\partial \lambda_{F_2}} = (-1)^2 \sum_{i_1=1}^{2} \sum_{\substack{i_2=i_1+1 \\ i_1,\, i_2 \neq 2}}^{3} \lambda_{F_{i_1}} \lambda_{F_{i_2}} = \lambda_{F_1} \lambda_{F_3}$$

$$\frac{\partial h_{12}}{\partial \lambda_{F_2}} = (-1)^1 \sum_{\substack{i_1=1 \\ i_1 \neq 2}}^{3} \lambda_{F_{i_1}} = -(\lambda_{F_1} + \lambda_{F_3})$$

$$\frac{\partial h_{13}}{\partial \lambda_{F_2}} = (-1)^0 = 1$$

$$\frac{\partial h_{14}}{\partial \lambda_{F_2}} \text{ to } \frac{\partial h_{1n}}{\partial \lambda_{F_2}} = 0$$

which is easily validated by taking the partials of Eqs (145) to (148) with respect to λ_{F_2}. Also

$$\frac{\partial \underline{H}^T}{\partial \underline{\lambda}_F} = \left[\frac{\partial \underline{H}^T}{\partial \lambda_{F_1}} \middle| \frac{\partial \underline{H}^T}{\partial \lambda_{F_2}} \middle| \cdots \middle| \frac{\partial \underline{H}^T}{\partial \lambda_{F_n}} \right] \tag{167}$$

Using Eqs (101), (163), (167), and the conjugate of Eq (165), $\frac{\partial M_m(\underline{F}_y)}{\partial \underline{\lambda}_F}$ can be obtained. This gives the direction that the feedback system eigenvalues should move in order to increase M. If equality constraints are imposed or the boundaries of inequality constraints are reached, the eigenvalues must move along the constraints while still maximizing M. The gradient and value of M can be evaluated at each step and a check made to ensure that M is indeed increasing.

To handle the equality and inequality constraints there are many methods that can be used (Ref 19). One method is to use the equality constraints to decrease the dimension of the problem. This is a very good method, but in general is difficult to apply if the constraint equations are complex and nonlinear. Another method is to void the constraints by means of penalty functions. This method adjoins the equality constraints to the cost function and weights each constraint. That is

$$\text{Cost} = M_m - \underline{\Psi}^T(\underline{\lambda}_F)\underline{\underline{\Xi}}\underline{\Psi}(\underline{\lambda}_F) \tag{168}$$

where $\underline{\Psi}(\underline{\lambda}_F)$ is the vector consisting of the constraints, and $\underline{\underline{\Xi}}$ is a positive definite weighting matrix, with iteratively increasing eigenvalues. Since the $\underline{\Psi}(\underline{\lambda}_F)$'s are supposed to be zero, the cost function penalizes any nonzero value. This approach, however, leads to a more difficult cost max-

imization problem.

The approach that is used is the gradient projection method. This method is useful for both equality and inequality constraints. The basic concept is that a step to change $\underline{\lambda}_F$ is made so that it not only increases the cost the most, to first order, but reduces the equality constraints towards zero and ignores inequality constraints if $\underline{\lambda}_F$ is in the correct region. Information on the gradient projection method can be found in Rosen's articles (Ref 30 and 31).

The inequality constraints that define Φ in general would be simple enough to handle directly. For example if Φ is such that only a certain degree of stability is required then Φ could be chosen as

$$\Phi = \left[\underline{\lambda}_F : \left|\lambda_{F_k}\right| \leq \bar{R},\ k = 1, 2, \cdots, n;\ \bar{R} < 1\right] \tag{169}$$

This is shown in Fig 4. The constraint boundary is well defined,

$$(\mathrm{Re}\ \lambda_{F_k})^2 + (\mathrm{Im}\ \lambda_{F_k})^2 = \bar{R}^2 \tag{170}$$

and it is not difficult to keep the eigenvalues on this boundary even when the gradient points out of the region.

With the information presented in the previous subsections, it is now possible to outline the algorithm that will provide the optimal $\underline{F}_y$ and $\underline{V}$ that maximizes the infor-

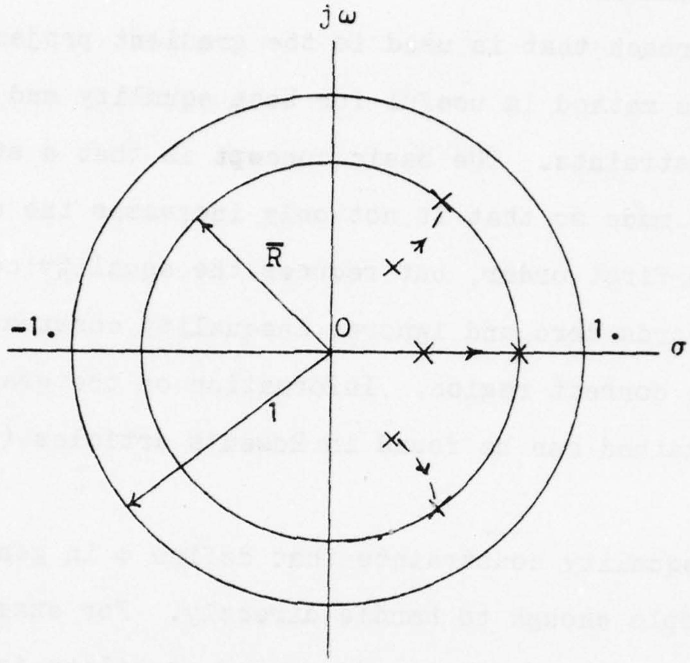
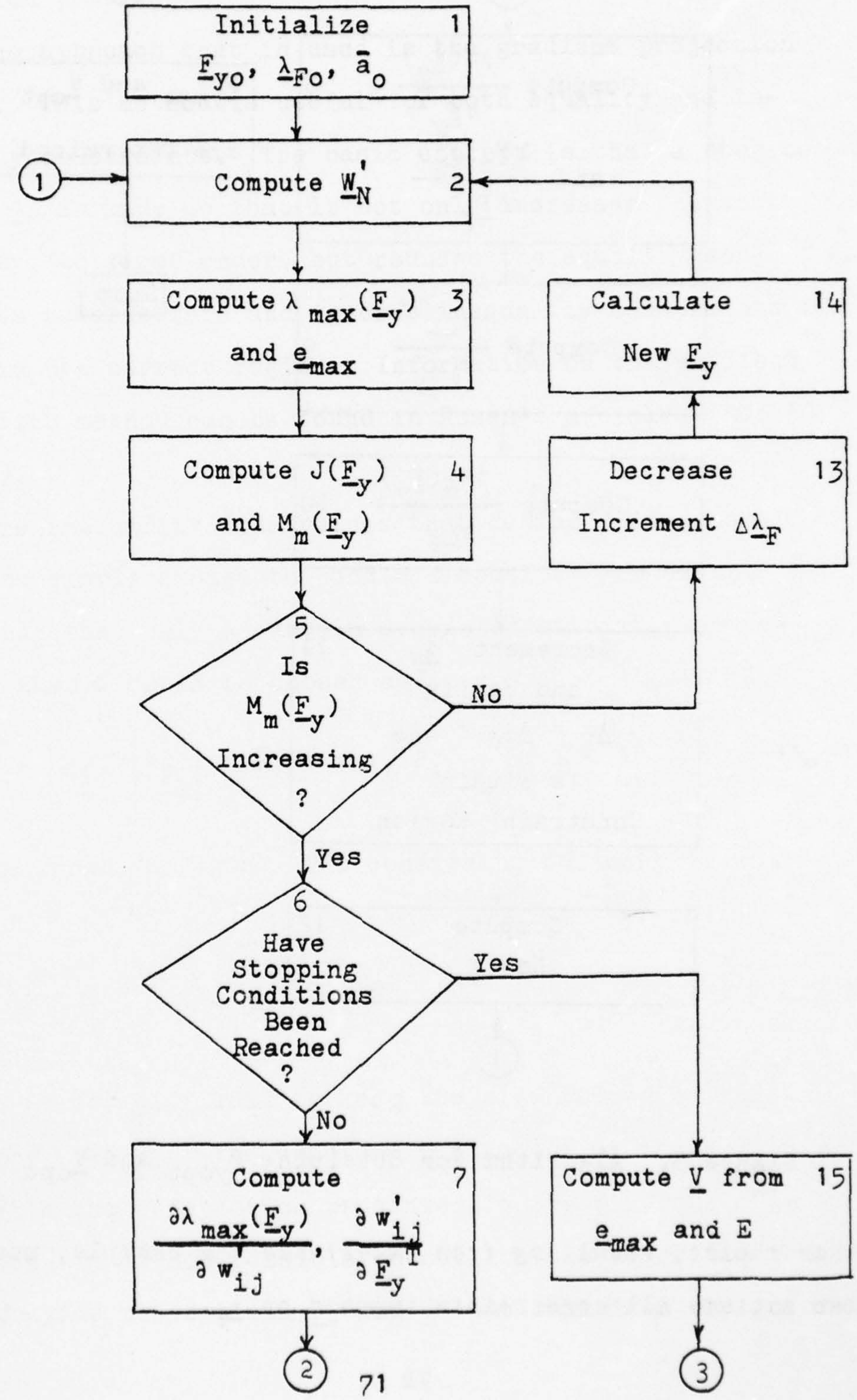


Figure 4. $\left|\lambda_{F_k}\right| \leq \overline{R}$, $\overline{R} < 1$, Eigenvalue Trajectories

mation.

Algorithm for Solution

The algorithm for finding the optimal feedback matrix and external controls is shown in Figure 5. The algorithm is initialized with the values $\underline{F}_{yo}$, $\underline{\lambda}_{Fo}$, and $\bar{a}_o$ as shown in block 1. The choices of $\underline{F}_{yo}$, $\underline{\lambda}_{Fo}$, and $\bar{a}_o$ are naturally not independent. The $\underline{F}_{yo}$ must be such that the eigenvalues of $\underline{A}(\bar{a}) + \underline{B}(\bar{a})\underline{F}_{yo}\underline{C}$ are $\underline{\lambda}_{Fo}$. An initial choice of $\underline{F}_{yo} = \underline{0}$ and $\underline{\lambda}_{Fo}$ being the eigenvalues of $\underline{A}(\bar{a})$ not only is a reasonable choice, but also guarantees that the equality constraints for consistency, Eqs (128) and (162), are satisfied. Any


Initialize 1
$\underline{F}_{yo}$, $\underline{\lambda}_{Fo}$, $\bar{a}_o$
1
Compute $\underline{W}_N'$ 2
Compute $\lambda_{max}(\underline{F}_y)$ 3
and $\underline{e}_{max}$
Compute $J(\underline{F}_y)$ 4
and $M_m(\underline{F}_y)$
5
Is
$M_m(\underline{F}_y)$
Increasing
?
No
Yes
6
Have
Stopping
Conditions
Been
Reached
?
Yes
No
Compute 7
$\frac{\partial \lambda_{max}(\underline{F}_y)}{\partial w_{ij}'}$, $\frac{\partial w_{ij}'}{\partial \underline{F}_y^T}$
Calculate 14
New $\underline{F}_y$
Decrease 13
Increment $\Delta\underline{\lambda}_F$
Compute $\underline{V}$ from 15
$\underline{e}_{max}$ and E
2
3

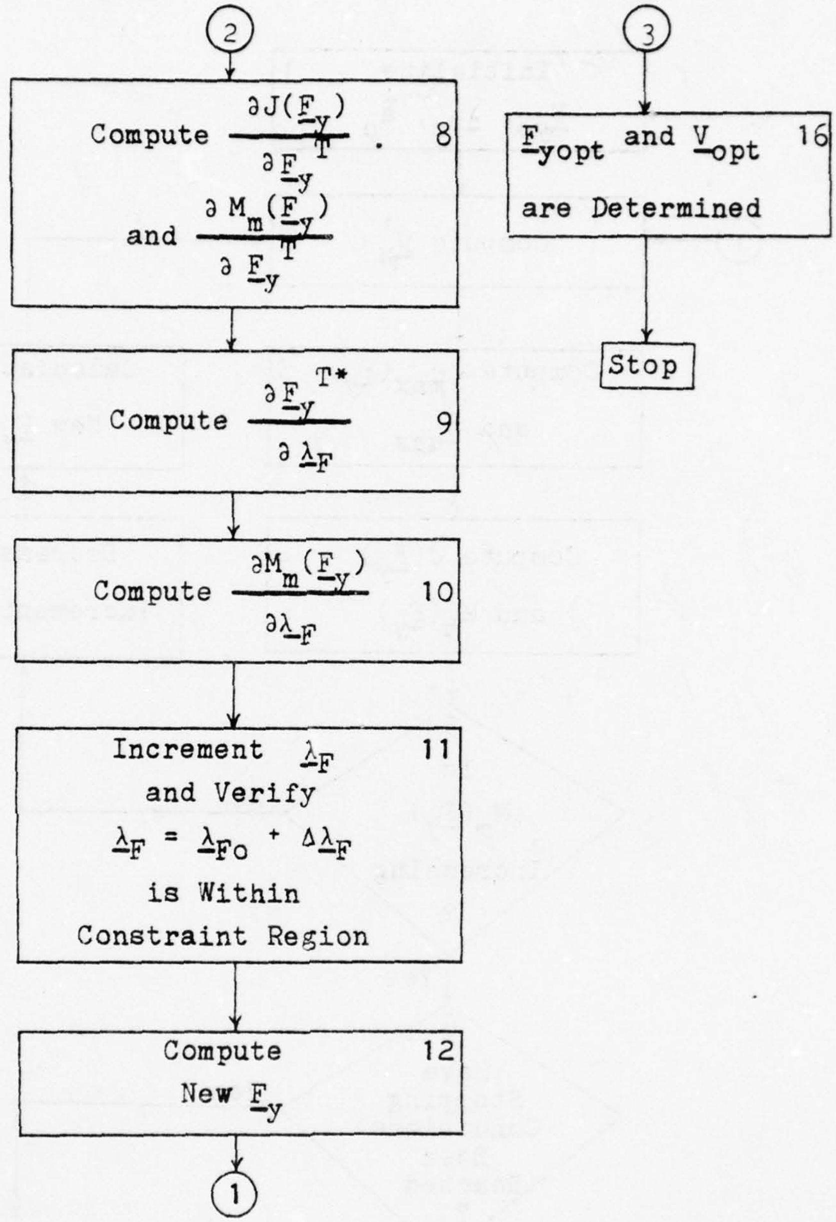


Figure 5. Algorithm For Obtaining $\underline{F}_{yopt}$ and $\underline{V}_{opt}$

other choice, resulting from experience for example, must also satisfy all constraints imposed on $\underline{\lambda}_F$.

With the initial values determined, block 2 shows that $\underline{W}_N'$ is computed from Eq (83). Computing $\lambda_{max}(\underline{F}_y)$ and $\underline{e}_{max}$ of $\underline{W}_N'$ as outlined in block 3 can be performed using many existing methods. The one used in this research was EIGRF subroutine which is part of the IMSL pack for the CDC-6600. Computing $J(\underline{F}_y)$ from Eq (86) and then using Eq (93), the value of $M_m(\underline{F}_y)$ is obtained. This is shown in block 4.

For the initial pass through the algorithm, the next step would be decision block 6. For other cases, decision block 5 is to check to see if $M_m(\underline{F}_y)$ had increased with the new $\underline{\lambda}_F$. It is possible that the incremental step taken, $\Delta\underline{\lambda}_F$, may be too large and cause the new $\underline{\lambda}_F$ to decrease the value of $M_m(\underline{F}_y)$. If this does occur, the procedure (Ref 19) is to decrease this increment step and repeat the previous steps. This is shown as blocks 13 and 14.

The stopping conditions are discussed later; for now it is assumed they have not been reached. Since the new $\underline{\lambda}_F$ has increased $M_m(\underline{F}_y)$ it is now necessary to select another step in $\underline{\lambda}_F$. The partials in blocks 7 and 8 are computed from Eqs (101), (103), (107), (113), and (120). In order to transfer this gradient to the eigenvalue space, $\dfrac{\partial \underline{F}_y^{T*}}{\partial \underline{\lambda}_F}$ must be computed, as referenced in block 9, by taking the complex conjugate of Eq (165). Using Eq (163) it is now possible to derive $\dfrac{\partial M_m(\underline{F}_y)}{\partial \underline{\lambda}_F}$. This is shown in block 10.

Knowing the gradient, the next step as shown in block 11 is to determine the increment step, $\Delta\underline{\lambda}_F$, that will be used to move $\underline{\lambda}_F$ in the gradient direction. There are many ways to determine the size of $\Delta\underline{\lambda}_F$, but the method found most acceptable in this research is to use a direct control on the step length. This means that $|\Delta\underline{\lambda}_F|$ is set but the direction depends on the gradient and constraints.

With $\Delta\underline{\lambda}_F$ determined, the new $\underline{\lambda}_F$ becomes:

$$\text{new } \underline{\lambda}_F = \text{old } \underline{\lambda}_F + \Delta\underline{\lambda}_F \tag{171}$$

As shown in block 12, the $\underline{F}_y$ that produces the new feedback system eigenvalues can be computed from Eqs (153) and (155). The complete procedure is repeated until a stopping condition is met.

The stopping conditions used in this research are

(1) When the change in $M_m(\underline{F}_y)$ is less than some preselected value, or

(2) When a time limit has been reached.

These were selected to limit the computation time while still giving good results. Ideally one would want to stop only when the maximum value of $M_m(\underline{F}_y)$ is reached; this is not practical since the optimal value of $M_m(\underline{F}_y)$ is not known.

Chapter III addresses an example and provides results that not only show the parameter estimate statistics, but compares this to the estimate statistics in the case of applying only optimal open-loop controls.

III Optimal Inputs for Three Dimensional Example

Introduction

In this chapter the developed algorithm is applied to an example. The optimal values of $\underline{F}_y$ and $\underline{V}$ are obtained for many different conditions. A maximum likelihood estimator is then designed and applied to the example to estimate the unknown parameter when the feedback controls are used. Results from the experiment are tabulated along with results when using an optimal open-loop control for the same example. The advantage of the feedback controls is especially obvious when the level of the measurement noise is very large.

Before presenting this example, the maximum likelihood estimator is developed. The estimator algorithm is presented along with equations that describe the estimator. Next the example problem is given. Using this example system the procedure to compute $\underline{F}_{yopt}$ and $\underline{V}_{opt}$ in Chapter II is used to obtain the optimal input controls. A nominal value for $\bar{a}$ is used in obtaining $\underline{F}_{yopt}$ and $\underline{V}_{opt}$. However, a different value of $\bar{a}$ is used when $\underline{F}_{yopt}$ and $\underline{V}_{opt}$ are applied to the system to get the output measurements. A noise generator is used to corrupt the measurement.

The input control sequence and the output measurements are then inputs to the estimator. Results of the estimates are tabulated and compared. Noise levels as well as the length of the control sequence $\underline{V}$ are varied for evaluation.

A two dimensional output system as well as a three dimensional output system is studied.

The case in which no feedback is used is also investigated. Using the optimal open-loop controls only provides a basis for comparison with the feedback system.

Maximum Likelihood Estimator

The likelihood function for this problem is $\ln p(Y_N | \bar{a})$. The purpose of the estimator is to find the parameter value $\bar{a}$ that maximizes the likelihood function. This is equivalent to finding the $\bar{a}$ that maximizes the probability that the observed measurement values resulted from the unknown parameter being $\bar{a}$. Using Eq (31) the likelihood function is

$$L'(\bar{a}) = \text{Constant} - \tfrac{1}{2}\sum_{j=1}^{N} \underline{\bar{n}}^T(j)\underline{R}^{-1}\underline{\bar{n}}(j) \tag{172}$$

Maximizing $L'(\bar{a})$ is the same as minimizing $L(\bar{a})$ where

$$L(\bar{a}) = \tfrac{1}{2}\sum_{j=1}^{N} (\underline{y}(j) - \underline{Cx}(j))^T\underline{R}^{-1}(\underline{y}(j) - \underline{Cx}(j)) \tag{173}$$

therefore,

$$L(\hat{\bar{a}}) = \min_{\bar{a}} L(\bar{a}) \tag{174}$$

To solve Eq (174) a combination gradient/Newton-Raphson method is used. The reason that the combination method is used is that the Newton-Raphson approach converges faster than gradient algorithms near the solution, but may

not converge far from the solution. The gradient method is guaranteed to converge to a solution, but converges slowly near the solution. The combination method uses the gradient approach to get near the solution and then switches to the Newton-Raphson approach to attain rapid final convergence.

The equation for incrementing $\bar{a}$ by the gradient method is

$$\bar{a}_{i+1} = \bar{a}_i - k_s \frac{\partial L(\bar{a}_i)}{\partial \bar{a}} \tag{175}$$

where $\bar{a}_i$ is the value of $\bar{a}$ from the i-th iteration, and k_s is the step size value that will be determined. The equation for the Newton-Raphson approach is

$$\bar{a}_{i+1} = \bar{a}_i - \left[\frac{\partial^2 L(\bar{a}_i)}{\partial \bar{a}^2}\right]^{-1} \left[\frac{\partial L(\bar{a}_i)}{\partial \bar{a}}\right] \tag{176}$$

Two points that need to be discussed before the estimator algorithm can be developed are

(1) How to determine k_s in Eq (175) and

(2) When will the switch from the gradient to the Newton-Raphson method take place?

Both answers depend on the problem to be solved. Many approaches (Ref 19) have been developed for finding k_s such as a direct constraint on the step length, constraining the predicted change in $L(\bar{a})$, and finding the minimum of the locally approximating parabola. The method chosen for this

example is the direct constraint on the step length, since the gradient method is used only to get close to the solution. The accuracy and complexity of the local parabola approach is not needed and the total change in $L(\bar{a})$ is unknown. Therefore, k_s is chosen to be

$$k_s = \frac{.015}{\left|\frac{\partial L(\bar{a})}{\partial \bar{a}}\right|} \tag{177}$$

The .015 was selected from experience to keep the number of iterations to a reasonable level. For this problem, the time for switching to Newton-Raphson will occur at the first time the function $L(\bar{a})$ starts to increase.

From Eqs (175) and (176) it can be seen that $\frac{\partial L(\bar{a})}{\partial \bar{a}}$ and $\frac{\partial^2 L(\bar{a})}{\partial \bar{a}^2}$ must be evaluated. From Eq (173)

$$\frac{\partial L(\bar{a})}{\partial \bar{a}} = - \sum_{j=1}^{N} \left[\underline{C}\frac{\partial \underline{x}(j)}{\partial \bar{a}}\right]^T \underline{R}^{-1}(\underline{y}(j) - \underline{C}\underline{x}(j)) \tag{178}$$

and

$$\frac{\partial^2 L(\bar{a})}{\partial \bar{a}^2} = - \sum_{j=1}^{N} \left[\left[\underline{C}\frac{\partial^2 \underline{x}(j)}{\partial \bar{a}^2}\right] \underline{R}^{-1}(\underline{y}(j) - \underline{C}\underline{x}(j)) - \left[\underline{C}\frac{\partial \underline{x}(j)}{\partial \bar{a}}\right]^T \underline{R}^{-1}\left[\underline{C}\frac{\partial \underline{x}(j)}{\partial \bar{a}}\right]\right] \tag{179}$$

To calculate $\frac{\partial \underline{x}(j)}{\partial \bar{a}}$ and $\frac{\partial^2 \underline{x}(j)}{\partial \bar{a}^2}$, Eq (14) is used.

$$\frac{\partial \underline{x}(j)}{\partial \bar{a}} = \frac{\partial \underline{A}(\bar{a})}{\partial \bar{a}}\underline{x}(j) + \underline{A}(\bar{a})\frac{\partial \underline{x}(j)}{\partial \bar{a}} + \frac{\partial \underline{B}(\bar{a})}{\partial \bar{a}}u(j) \tag{180}$$

and

$$\frac{\partial^2 \underline{x}(j)}{\partial \bar{a}^2} = \frac{\partial^2 \underline{A}(\bar{a})}{\partial \bar{a}^2}\underline{x}(j) + 2\frac{\partial \underline{A}(\bar{a})}{\partial \bar{a}}\,\frac{\partial \underline{x}(j)}{\partial \bar{a}}$$

$$+ \underline{A}(a)\frac{\partial^2 \underline{x}(j)}{\partial \bar{a}^2} + \frac{\partial^2 \underline{B}(\bar{a})}{\partial \bar{a}^2}u(j) \tag{181}$$

where

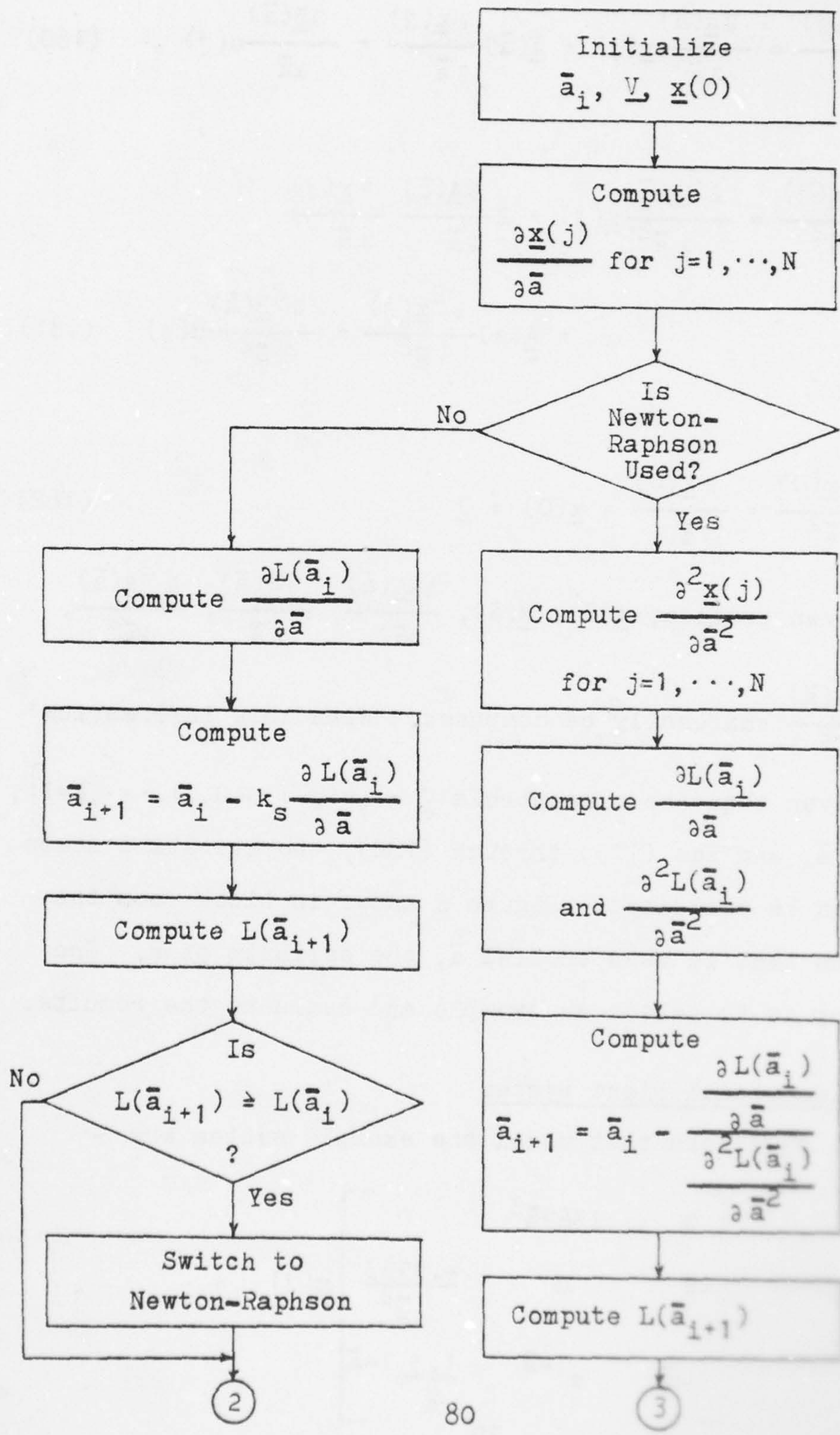
$$\frac{\partial^2 \underline{x}(0)}{\partial \bar{a}^2} = \frac{\partial \underline{x}(0)}{\partial \bar{a}} = \underline{x}(0) = \underline{0} \tag{182}$$

For a given problem, $\underline{A}(\bar{a})$, $\underline{B}(\bar{a})$, $\frac{\partial \underline{A}(\bar{a})}{\partial \bar{a}}$, $\frac{\partial \underline{B}(\bar{a})}{\partial \bar{a}}$, $\frac{\partial^2 \underline{A}(\bar{a})}{\partial \bar{a}^2}$, and $\frac{\partial^2 \underline{B}(\bar{a})}{\partial \bar{a}^2}$ can easily be computed. With this information' and a given sequence of controls $\underline{V} = \left[v(j): j=0, 1,\cdots,N-1\right]$, a given $\bar{a}$, and Eqs (175) through (182), the iterative steps for $\bar{a}$ can be evaluated. Figure 6 shows in block form the algorithm that is used to find $\hat{\bar{a}}$, the estimate of $\bar{a}$. The next step is to select an example and evaluate the results.

Three Dimensional Plant System

The equations that model the example system are

$$\underline{x}(j+1) = \begin{bmatrix} 0 & .1367\bar{a}^3 & 0 \\ -\bar{a} & 0 & \frac{-.7733}{\bar{a}^2} \\ 1-\bar{a} & e^{1-\bar{a}} & \frac{1.5}{\bar{a}}e^{1-\bar{a}} \end{bmatrix}\underline{x}(j) \quad +$$


Initialize
$\bar{a}_i$, $\underline{V}$, $\underline{x}(0)$
Compute
$\frac{\partial \underline{x}(j)}{\partial \bar{a}}$ for $j=1,\cdots,N$
1
Is Newton-Raphson Used?
No
Yes
Compute $\frac{\partial L(\bar{a}_i)}{\partial \bar{a}}$
Compute
$\bar{a}_{i+1} = \bar{a}_i - k_s \frac{\partial L(\bar{a}_i)}{\partial \bar{a}}$
Compute $L(\bar{a}_{i+1})$
Is $L(\bar{a}_{i+1}) \geq L(\bar{a}_i)$?
No
Yes
Switch to Newton-Raphson
2
Compute $\frac{\partial^2 \underline{x}(j)}{\partial \bar{a}^2}$
for $j=1,\cdots,N$
Compute $\frac{\partial L(\bar{a}_i)}{\partial \bar{a}}$
and $\frac{\partial^2 L(\bar{a}_i)}{\partial \bar{a}^2}$
Compute
$\bar{a}_{i+1} = \bar{a}_i - \frac{\frac{\partial L(\bar{a}_i)}{\partial \bar{a}}}{\frac{\partial^2 L(\bar{a}_i)}{\partial \bar{a}^2}}$
Compute $L(\bar{a}_{i+1})$
3

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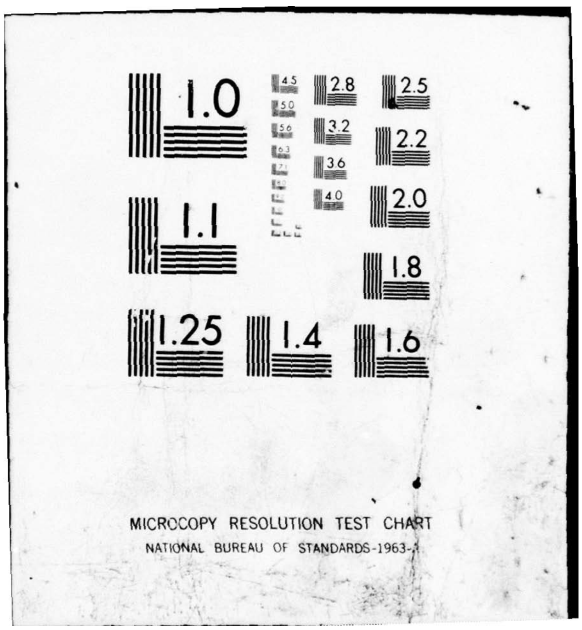
1.0
1.1
1.25
1.4
1.6
1.8
2.0
2.2
2.5
2.8
3.2
3.6
4.0
MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

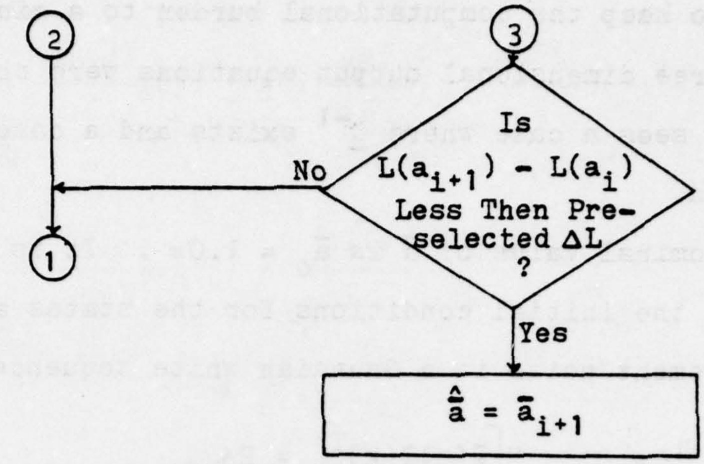

Figure 6. Estimator Algorithm

$$+ \begin{bmatrix} 1-\bar{a} \\ 0 \\ e^{2(1-\bar{a})} \end{bmatrix} u(j) \qquad (183)$$

and

$$\underline{y}(j) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \underline{x}(j) + \underline{\bar{w}}(j) \qquad (184)$$

for the two dimensional output system and

$$\underline{y}(j) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underline{x}(j) + \underline{\bar{w}}(j) \qquad (185)$$

for the three dimensional output system. This particular form was chosen for two reasons. First, the system is not linear in the parameter and second, the numbers were selected so that the consistency constraint is fairly simple

in order to keep the computational burden to a minimum. The two and three dimensional output equations were chosen so the reader sees a case where $\underline{C}^{-1}$ exists and a case where $\underline{C}^{+}$ is required.

The nominal value of $\bar{a}$ is $\bar{a}_o = 1.0$. It is also assumed that the initial conditions for the states are zero. The measurement noise is a Gaussian white sequence with

$$E\left[\underline{\bar{w}}(j)\right] = \underline{0}; \quad E\left[\underline{\bar{w}}(j)\underline{\bar{w}}(k)^T\right] = \underline{R}\delta_{kj} \tag{186}$$

and a Gaussian noise generator is used in the example. The covariance matrix $\underline{R}$ is varied for different runs of the experiment.

From Eq (183)

$$\frac{\partial \underline{A}(\bar{a})}{\partial \bar{a}} = \begin{bmatrix} 0 & .41\bar{a}^2 & 0 \\ -1 & 0 & \dfrac{1.5466}{\bar{a}^3} \\ -1 & -e^{1-\bar{a}} & -\left[\dfrac{1.5}{\bar{a}} + \dfrac{1.5}{\bar{a}^2}\right]e^{1-\bar{a}} \end{bmatrix} \tag{187}$$

$$\frac{\partial \underline{B}(\bar{a})}{\partial \bar{a}} = \begin{bmatrix} -1 \\ 0 \\ -2e^{2(1-\bar{a})} \end{bmatrix} \tag{188}$$

$\dfrac{\partial^2 \underline{A}(\bar{a})}{\partial \bar{a}^2}$ and $\dfrac{\partial^2 \underline{B}(\bar{a})}{\partial \bar{a}^2}$ are then derived from Eqs (187) and (188).

Obtaining Estimator Inputs

Inputs that are required by the estimator are the control sequence $\underline{V}$ and measurements made on the system's output. The external control sequence $\underline{V}$ will be $\underline{V}_{opt}$ and will be computed using the algorithm developed in Chapter II. The measurements will be obtained when $\underline{V}_{opt}$ and $\underline{F}_{yopt}$ are applied to the system with the true $\bar{a}$, $\bar{a}_T = 1.2$, while the $\bar{a}$ used in the filter is $\bar{a}_o = 1.0$, and the output is corrupted by a white Gaussian noise.

For all cases the energy constraint on the external controls is unity. The length of the control sequence is varied from two to eight steps. The values for $\underline{R}$ are varied; however, the $\underline{R}$ used to compute $\underline{F}_{yopt}$ and $\underline{V}_{opt}$ is the same $\underline{R}$ used to generate the output measurements when $\underline{F}_{yopt}$ and $\underline{V}_{opt}$ are applied to the system.

There are constraints on the eigenvalues in order to give the desired stability and time response characteristics. These constraints are

$$\left|\lambda_{F_k}\right| < .9 \tag{189}$$

$$\mathrm{Re}\,\lambda_{F_k} \geq 0 \tag{190}$$

$$\mathrm{Imag}\,\lambda_{F_k} \leq \mathrm{Re}\,\lambda_{F_k} \tag{191}$$

These inequality constraints determine the constraint space Φ. For the two dimensional output example it can easily be seen that the rank of $\underline{C}$ is less than the number of states.

This implies that an equality constraint on $\underline{\lambda}_F$ is needed. Only one equality constraint is needed since the number of states is three and the rank of $\underline{C}$ is two. The constraint will be computed by using the equations developed in the previous chapter and in Appendix F which the reader should read to understand the following example.

With $\bar{\bar{a}} = 1.0$, the system equations become:

$$\underline{x}(j+1) = \begin{bmatrix} 0 & .13667 & 0 \\ -1 & 0 & -.7733 \\ 0 & 1 & 1.5 \end{bmatrix} \underline{x}(j) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(j) \qquad (192)$$

$$\underline{y}(j) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \underline{x}(j) + \underline{\bar{w}}(j) \qquad (193)$$

Since the controllability matrix

$$\underline{\bar{Q}} = \left[\underline{B}(\bar{a}_o), \underline{A}(\bar{a}_o)\underline{B}(\bar{a}_o), \underline{A}^2(\bar{a}_o)\underline{B}(\bar{a}_o) \right]$$

$$\underline{\bar{Q}} = \begin{bmatrix} 0 & 0 & -.1057 \\ 0 & -.7733 & -1.160 \\ 1 & 1.5 & 1.477 \end{bmatrix} \qquad (194)$$

which is rank three so as shown in Appendix F, $\underline{\hat{Q}} = \underline{\bar{Q}}$. Forming $\underline{\hat{Q}}^{-1}$ gives

$$\underline{\hat{Q}}^{-1} = \begin{bmatrix} -7.317 & 1.94 & 1.0 \\ 14.2 & -1.294 & 0 \\ -9.465 & 0 & 0 \end{bmatrix} \qquad (195)$$

Since $\underline{e}_{31} = \begin{bmatrix} -9.465 & 0 & 0 \end{bmatrix}$

$$\underline{T}^{-1} = \begin{bmatrix} -9.465 & 0 & 0 \\ 0 & -1.294 & 0 \\ 1.294 & 0 & 1.0 \end{bmatrix} \tag{196}$$

and

$$\underline{T} = \begin{bmatrix} -.1057 & 0 & 0 \\ 0 & -.7728 & 0 \\ .1367 & 0 & 1.0 \end{bmatrix} \tag{197}$$

Since $\underline{\bar{F}}_c = \underline{T}^{-1}\underline{A}(\bar{a}_o)\underline{T}$ and $\underline{\bar{G}}_c = \underline{T}^{-1}\underline{B}(\bar{a}_o)$

$$\underline{\bar{F}}_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ .205 & -.91 & 1.5 \end{bmatrix} \tag{198}$$

$$\underline{\bar{G}}_c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{199}$$

$\underline{F}_c = \underline{\bar{F}}_c - \underline{\bar{G}}_c\underline{L}$ and $\underline{G}_c = \underline{\bar{G}}_c$ since $\underline{Z} = 1$ for this example.
Then since $\underline{L} = \begin{bmatrix} l_1, & l_2, & l_3 \end{bmatrix}$

$$\underline{F}_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ .205 & -.91 & 1.5 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ l_1 & l_2 & l_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{200}$$

so $\underline{L} = \begin{bmatrix} .205 & -.91 & 1.5 \end{bmatrix}$. Using Eq (142)

$$\underline{H} = \Big[\lambda_{F_1}\lambda_{F_2}\lambda_{F_3},\ -(\lambda_{F_1}\lambda_{F_2} + \lambda_{F_1}\lambda_{F_3} + \lambda_{F_2}\lambda_{F_3}),\ (\lambda_{F_1} + \lambda_{F_2} + \lambda_{F_3})\Big] \tag{201}$$

From Eq (153) the state feedback matrix is

$$\underline{F}_{\epsilon} = \Big[(\lambda_{F_1}\lambda_{F_2}\lambda_{F_3} - .205),\ (.91 - \lambda_{F_1}\lambda_{F_2} - \lambda_{F_1}\lambda_{F_3} - \lambda_{F_2}\lambda_{F_3}),\ (\lambda_{F_1} + \lambda_{F_2} + \lambda_{F_3} - 1.5)\Big]\underline{T}^{-1} \tag{202}$$

Using Eq (162),

$$f_{s_3} = k_{31}f_{s_1} + k_{32}f_{s_2} \tag{203}$$

and from $\underline{C}$ it is clear that

$$k_{31} = k_{32} = 0 \tag{204}$$

so

$$f_{s_3} = 0 \tag{205}$$

Then from Eqs (196) and (202)

$$f_{s_3} = \lambda_{F_1} + \lambda_{F_2} + \lambda_{F_3} - 1.5 \tag{206}$$

Combining Eqs (205) and (206) implies that the equality constraint needed to force consistency is

$$\lambda_{F_1} + \lambda_{F_2} + \lambda_{F_3} = 1.5 \tag{207}$$

With this constraint on the eigenvalues then

$$\underline{F}_y = \underline{F}_s \underline{C}^T (\underline{C}\underline{C}^T)^{-1} \tag{155}$$

will give the desired output feedback matrix. Substituting the value of $\underline{C}$, $\underline{T}^{-1}$, and Eq (202) into Eq (155) will give $\underline{F}_y$ in terms of $\underline{\lambda}_F$.

The algorithm in Figure 5 was used and part of the results are tabulated in Tables I and II. The tables give

Table I. $\underline{V}_{opt}$ When N = 8

Time	Feedback		No Feedback	
	R = .01	R = 10.	R = .01	R = 10.
0	.625	.640	.378	.378
1	.534	.537	.503	.503
2	.425	.418	.529	.529
3	.308	.294	.450	.450
4	.194	.181	.307	.307
5	.100	.090	.159	.159
6	.036	.031	.051	.051
7	-.001	-.001	-.002	-.002

only the optimal open-loop, $\underline{V}_{opt}$, portion of the optimal feedback control along with the optimal open-loop control if there were no feedback. Table I shows both controls for the case when N = 8 and the measurement noise levels $\underline{R} = R\underline{I}$ are different. Table II gives the open-loop control sequences for cases when the total number of time steps vary

Table II. $\underline{V}_{opt}$ For Two Dimensional Output

	Number of Steps (N)											
	Feedback						No Feedback					
Time	2	3	4	5	6	7	2	3	4	5	6	7
0	1.0	.927	.861	.799	.742	.688	1.0	.908	.812	.709	.597	.484
1	0	.375	.474	.522	.542	.545	0	.418	.538	.588	.593	.561
2		.001	.182	.273	.346	.391		-.003	.227	.364	.459	.515
3			-.001	.102	.180	.244			-.002	.142	.267	.373
4				-.001	.064	.124				-.002	.098	.204
5					-.001	.048					-.002	.070
6						-.001						-.002

$\underline{R} = R\underline{I}$ where $R = 1$.

from two to seven. It can be seen from the tables that with feedback, most of the energy is applied earlier in the sequence. The system knows that the feedback controls will provide energy later in the sequence. For the examples used about 94% of the total energy came from the feedback controls while about 6% came from the open-loop controls.

The optimal feedback matrix, $\underline{F}_{yopt}$, for all cases except when both the control sequence consisted of eight controls and the diagonal elements of $\underline{R}$ were .01 is

$$\underline{F}_{yopt} = \begin{bmatrix} 1.940, & -.449 \end{bmatrix} \tag{208}$$

For the case with the eight step control sequence

$$\underline{F}_{yopt} = \begin{bmatrix} 1.173, & -.362 \end{bmatrix} \tag{209}$$

Figures 7 and 8 show the trajectory of the eigenvalues while M_m is being optimized. Double poles occur at .75 and .3 respectively because of the constraint that the sum of the eigenvalues must be equal to 1.5.

For the three dimensional output example, no equality constraint is required since the rank of $\underline{C}$ is three. Table III lists $\underline{V}_{opt}$ for the feedback and open-loop cases where the control sequence is set at six steps. The optimal feedback matrix is

$$\underline{F}_{yopt} = \begin{bmatrix} -3.406, & 1.966, & 1.200 \end{bmatrix} \tag{210}$$

The eigenvalue trajectory is shown in Figure 9.

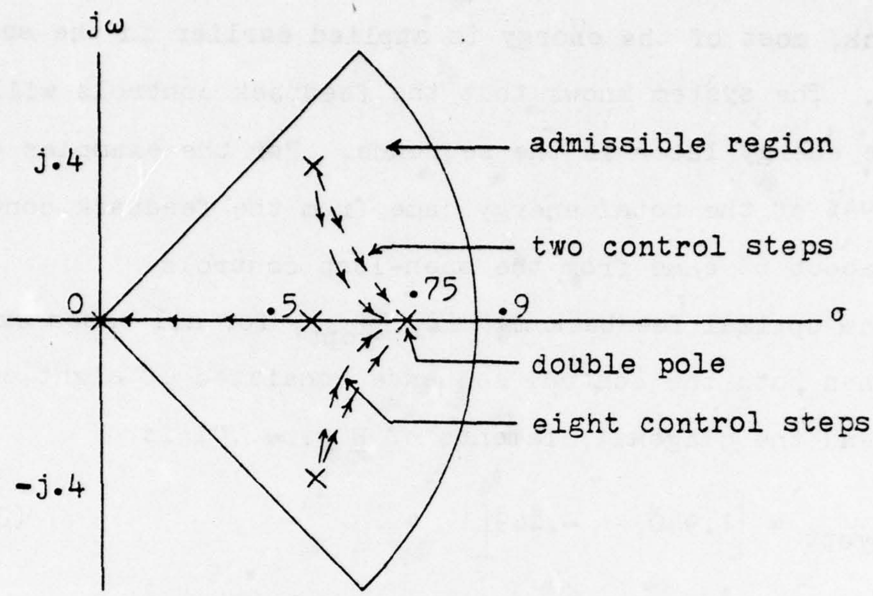

Figure 7. Trajectories for $\underline{F}_{yopt} = \begin{bmatrix} 1.940, & -.449 \end{bmatrix}$

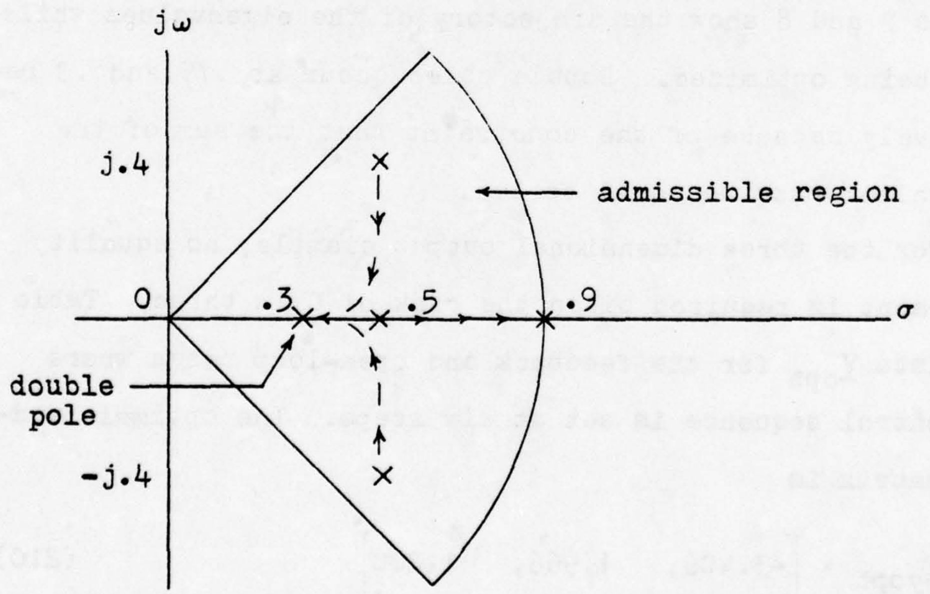

Figure 8. Trajectories for $\underline{F}_{yopt} = \begin{bmatrix} 1.173, & -.362 \end{bmatrix}$

Table III. $\underline{V}_{opt}$ For Three Dimensional Output

Time	Feedback N = 6	No Feedback N = 6
0	.756	.558
1	.564	.588
2	.309	.483
3	.120	.303
4	.023	.131
5	-.004	.019

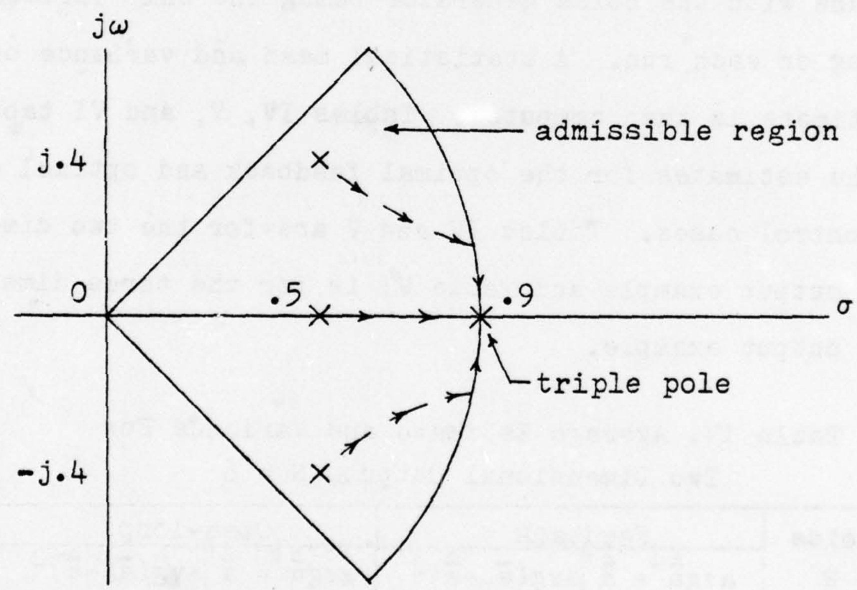

Figure 9. Trajectories for Three Dimensional Output Case

The feedback matrices and optimal external controls are used along with a Gaussian generator to produce measurements at each time step. The equations used are

$$\underline{x}(j+1) = \underline{A}(\bar{a}_T) + \underline{B}(\bar{a}_T)\underline{F}_{yopt}\underline{C}\ \underline{x}(j) + \underline{B}(\bar{a}_T)v(j) + \underline{B}(\bar{a}_T)\underline{F}_{yopt}\underline{\bar{w}}(j) \tag{211}$$

$$\underline{y}(j) = \underline{Cx}(j) + \underline{\bar{w}}(j) \tag{212}$$

where $\bar{a}_T$ is the true value of $\bar{a}$ which is 1.2. The actual measurements along with the controls are used in the algorithm of Figure 6 to estimate the unknown parameter and the next section discusses the results.

Estimate Statistics

A Monte Carlo analysis is used, composed of 50 simulation runs with the noise generator being the only variable changing on each run. A statistical mean and variance of the estimate is then computed. Tables IV, V, and VI tabulate the estimates for the optimal feedback and optimal open-loop control cases. Tables IV and V are for the two dimensional output example and Table VI is for the three dimensional output example.

Table IV. Average Estimate and Variance For Two Dimensional Output, N = 6

Noise R	Feedback		Open-loop	
	$\text{avg}\hat{\bar{a}} = \hat{\bar{a}}'$	$\text{avg}(\bar{a}_T-\hat{\bar{a}})^2$	$\text{avg}\hat{\bar{a}} = \hat{\bar{a}}'$	$\text{avg}(\bar{a}_T-\hat{\bar{a}})^2$
.01	1.2045	.00060	1.2026	.00037
.10	1.2001	.00474	1.2157	.00477
1.0	1.2018	.00729	1.2851	.06014
5.0	1.1993	.00819	1.3130	.12068
10.	1.1995	.00843	1.3191	.15053

$\bar{a}_T = 1.2$

Table V. Average Estimate and Variance For Feedback and Open-loop Controls

Control Steps N	Noise R	Feedback $\text{avg}\hat{\bar{a}} = \hat{\bar{a}}'$	Feedback $\text{avg}(\bar{a}_T - \hat{\bar{a}})^2$	Open-loop $\text{avg}\hat{\bar{a}} = \hat{\bar{a}}'$	Open-loop $\text{avg}(\bar{a}_T - \hat{\bar{a}})^2$
3	.01	1.20521	.00102	1.20184	.00058
	10.	1.18751	.03749	1.30675	.21327
4	.01	1.20593	.00101	1.20352	.00060
	10.	1.20610	.01561	1.30323	.14096
5	.01	1.20974	.00121	1.20575	.00054
	10.	1.19868	.00956	1.26783	.10304
7	.01	1.20215	.00065	1.20273	.00041
	10.	1.19839	.00572	1.33588	.14043
8	.01	1.20346	.00042	1.20251	.00040
	10.	1.20312	.00564	1.32748	.12040

$\underline{R} = R\underline{I}$; $\bar{a}_T = 1.2$

Table VI. Average Estimate and Variance For Three Dimensional Output, N = 6

Noise R	Feedback		Open-loop	
	$\text{avg}\hat{\bar{a}} = \hat{\bar{a}}'$	$\text{avg}(\bar{a}_T - \hat{\bar{a}})^2$	$\text{avg}\hat{\bar{a}} = \hat{\bar{a}}'$	$\text{avg}(\bar{a}_T - \hat{\bar{a}})^2$
.10	1.2004	.00019	1.1988	.00166
1.0	1.1985	.00030	1.2403	.03013
10.	1.2016	.00021	1.2895	.10035
25.	1.2018	.00023	1.2833	.12116

$\bar{a}_T = 1.2$

Table IV shows that with feedback the estimate of the parameter does not vary much for different levels of noise, but the confidence is higher for the cases with low noise. For the open-loop case, both the parameter estimate and confidence level worsen as the noise level increases. This is an important point; there is a significant performance benefit in using feedback over open-loop controls when $\underline{R}$ is large.

Table V shows that as the number of control steps increases, the parameter estimates improve and so do the confidence levels in these estimates for the feedback control example. For the open-loop case, this improvement is not obvious. As displayed in the table, the increased number of steps should improve the estimate, since more data is available for the estimator and more inputs are allowed to help the estimator. As should be expected, the table also shows that the estimates are better for the lower noise levels.

Table VI basically displays the same results as Table IV except that it can be seen that the confidence levels are better mainly due to the fact that more output data is fed back to the input.

Figures 10 and 11 present the information derived from Tables IV and VI in graphical form. It is seen that the

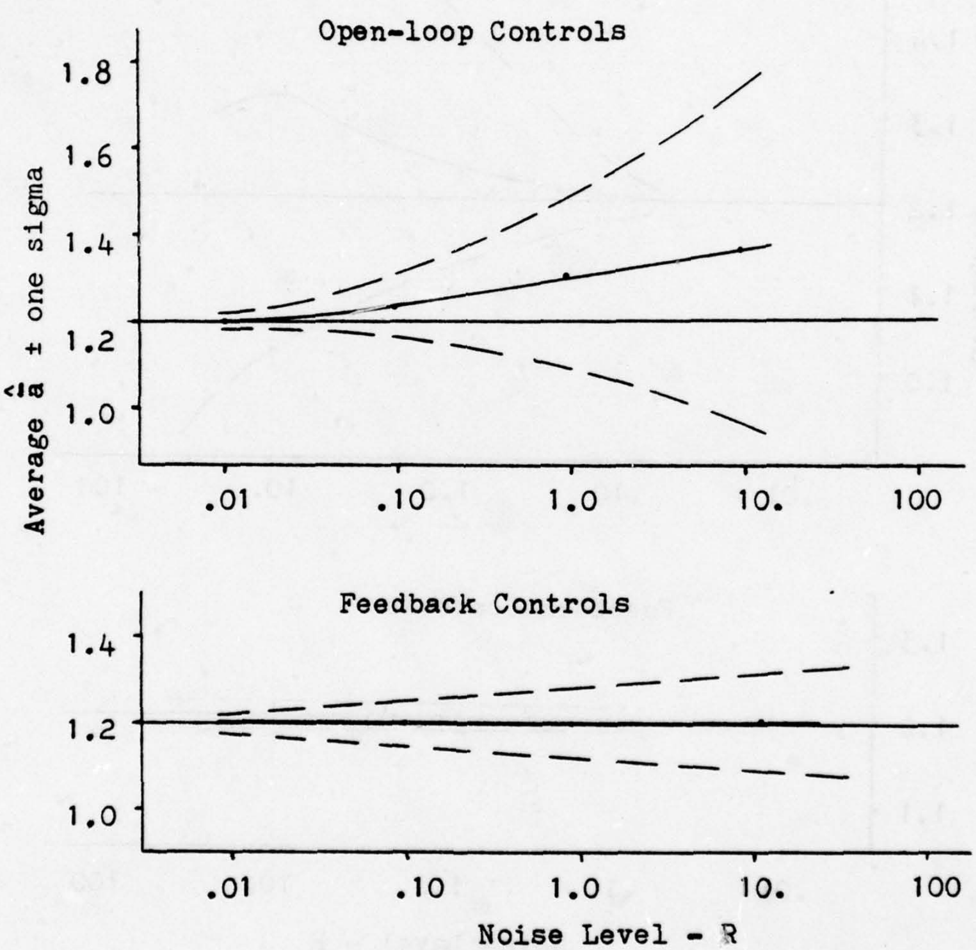

Figure 10. Average Estimate ± One Standard Deviation Vs Noise Level For The Two Dimensional Output Example

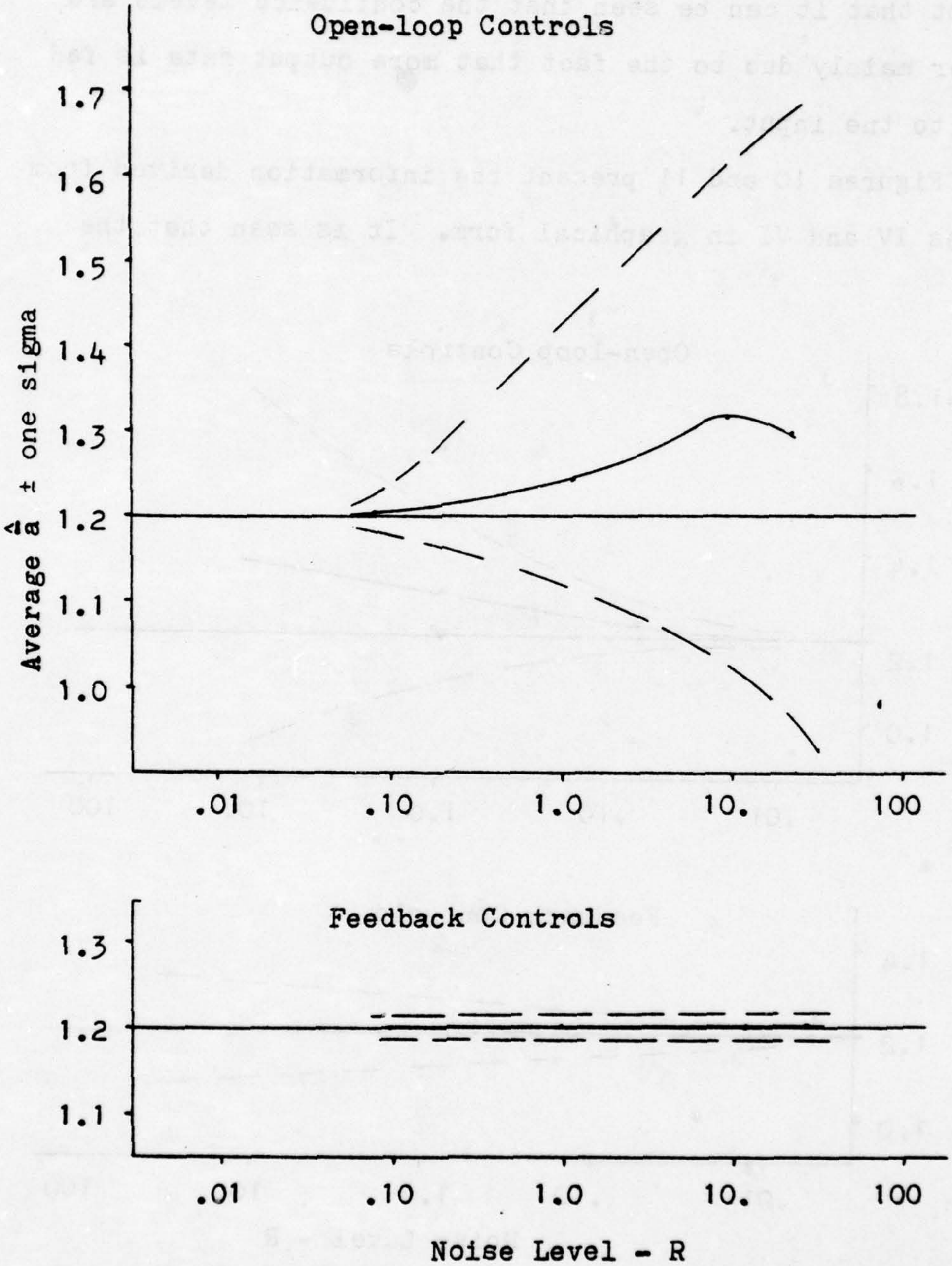

Figure 11. Average Estimate ± One Standard Deviation Vs Noise Level For The Three Dimensional Output Example

feedback controls improve the estimates considerably when there is much measurement noise present. For low noise levels the optimal feedback and open-loop controls produce about the same results. If there is no noise, both controls should produce a nearly perfect estimate. Figure 10 shows the average estimate with a ± one sigma deviation about it for the two dimensional output example and Figure 11 is the same for the three dimensional example.

Besides the fact that the more accurate estimates and variances are in general better for the feedback controls, another performance comparison is the computer execution time required by the estimator. All computation was done on a CDC 6600 computer. Table VII tabulates this information and clearly shows the time saved for cases with high noise levels when feedback is used. It is understood that just giving the computation times is not conclusive that one system is superior to another. Additions and multiplications required per method would be better as well as the number of iterations required. However, total time at least gives an idea of the relative complexity of the two approaches.

Conclusion

The results of the previous tables and figures show that for most cases, the feedback controls aided the estimator much more than the optimal open-loop controls. As was mentioned, if there is no noise present, then both systems should give the same results. As the noise level in-

Table VII. Computer Execution Time Required by Estimator

Control Steps N	Noise R	CP Seconds Exec. Time Feedback	Open-loop
3	.01	6.582	6.312
	10.	6.791	10.709
4	.01	8.751	8.693
	10.	8.522	12.788
5	.01	10.878	10.849
	10.	10.226	13.399
6	.01	12.922	12.876
	10.	12.891	19.201
7	.01	15.011	15.031
	10.	14.429	22.737
8	.01	17.063	16.883
	10.	16.494	24.992

creases, the feedback method shows a significant improvement. For small noise values it can be seen that in some cases the open-loop controls actually resulted in a better estimate; however, the feedback case was very close and the accuracy lost by using feedback control was small.

IV Multi-parameter and Multi-input Cases

Introduction

In Chapter II the problem of using feedback controls to aid the identification of a linear system model was solved for the case of a single unknown parameter and a single input control. The purpose was to simplify the problem so that a better understanding of the approach to the solution and development of the algorithm could be gained. It is shown in this chapter that, for the more complex problems, the basic approach taken for the single parameter, single input system can be used.

The next section will address the more realistic problem of how to compute the feedback gains and external open-loop controls when there is more than one unknown parameter. The following section looks at the case in which multi-inputs are required. Both cases increase the computational burden, and it may be practical to compute input controls only for small dimensional systems.

Multiple Unknown Parameters

With multiple unknown parameters, the information matrix of Eq (22) becomes:

$$\underline{M} = \mathop{E}_{Y_N}\left[\left[\frac{\partial \ln p(Y_N|\bar{\underline{a}})}{\partial \bar{\underline{a}}}\right]^T\left[\frac{\partial \ln p(Y_N|\bar{\underline{a}})}{\partial \bar{\underline{a}}}\right]\right] \tag{213}$$

where $\bar{\underline{a}}$ is a p-dimensional vector of unknown parameters.

From Eq (32)

$$\frac{\partial \ln p(Y_N|\underline{\bar{a}})}{\partial \underline{\bar{a}}} = - \sum_{j=1}^{N} \underline{\bar{n}}^T(j)\underline{R}^{-1}\left[\frac{\partial \underline{\bar{n}}(j)}{\partial \bar{a}_1}\,\middle|\,\cdots\,\middle|\,\frac{\partial \underline{\bar{n}}(j)}{\partial \bar{a}_p}\right] \tag{214}$$

therefore,

$$\underline{M} = \underset{Y_N}{E}\left\{\left(\sum_{j=1}^{N}\begin{bmatrix} \partial \underline{\bar{n}}^T(j)/\partial \bar{a}_1 \\ \hline \partial \underline{\bar{n}}^T(j)/\partial \bar{a}_2 \\ \hline \vdots \\ \hline \partial \underline{\bar{n}}^T(j)/\partial \bar{a}_p \end{bmatrix}\underline{R}^{-1}\underline{\bar{n}}(j)\right)\right.$$

$$\left.\cdot\left(\sum_{j=1}^{N}\underline{\bar{n}}^T(j)\underline{R}^{-1}\left[\frac{\partial \underline{\bar{n}}(j)}{\partial \bar{a}_1}\,\middle|\,\frac{\partial \underline{\bar{n}}(j)}{\partial \bar{a}_2}\,\middle|\cdots\middle|\,\frac{\partial \underline{\bar{n}}(j)}{\partial \bar{a}_p}\right]\right)\right\} \tag{215}$$

For the same reason as outlined in Chapter II, Eq (215) becomes:

$$\underline{M} = \underset{Y_N}{E}\left\{\sum_{j=1}^{N}\begin{bmatrix} \partial \underline{\bar{n}}^T(j)/\partial \bar{a}_1 \\ \hline \partial \underline{\bar{n}}^T(j)/\partial \bar{a}_2 \\ \hline \vdots \\ \hline \partial \underline{\bar{n}}^T(j)/\partial \bar{a}_p \end{bmatrix}\underline{R}^{-1}\underline{\bar{n}}(j)\underline{\bar{n}}^T(j)\underline{R}^{-1}\right.$$

$$\left.\cdot\left[\frac{\partial \underline{\bar{n}}(j)}{\partial \bar{a}_1}\,\middle|\,\frac{\partial \underline{\bar{n}}(j)}{\partial \bar{a}_2}\,\middle|\cdots\middle|\,\frac{\partial \underline{\bar{n}}(j)}{\partial \bar{a}_p}\right]\right\} \tag{216}$$

and it can be seen from Eqs (34) to (42) that the elements of $\underline{M}$ become:

$$m_{ik} = \sum_{j=1}^{N} \mathrm{tr}\left[E\left[\frac{\partial \underline{x}(j)}{\partial \bar{a}_i} \frac{\partial \underline{x}^T(j)}{\partial \bar{a}_k} \right] \underline{C}^T \underline{R}^{-1} \underline{C} \right] \tag{217}$$

Let

$$\underline{X}_{A_i}^T(k+1) = \left[\underline{x}^T(k+1), \frac{\partial \underline{x}^T(k+1)}{\partial \bar{a}_i} \right] \tag{218}$$

$$\underline{B}_{A_i}^T = \left[\underline{B}^T(\bar{\underline{a}}), \frac{\partial \underline{B}^T(\bar{\underline{a}})}{\partial \bar{a}_i} \right] \tag{219}$$

$$\underline{D}_{A_i}^T = \left[\underline{F}_y{}^T \underline{B}^T(\bar{\underline{a}}), \underline{F}_y{}^T \frac{\partial \underline{B}^T(\bar{\underline{a}})}{\partial \bar{a}_i} \right] \tag{220}$$

$$\underline{A}_{A_i} = \left[\begin{array}{c|c} \underline{A}(\bar{\underline{a}}) + \underline{B}(\bar{\underline{a}})\underline{F}_y\underline{C} & \underline{O} \\ \hline \frac{\partial \underline{A}(\bar{\underline{a}})}{\partial \bar{a}_i} + \frac{\partial \underline{B}(\bar{\underline{a}})}{\partial \bar{a}_i}\underline{F}_y\underline{C} & \underline{A}(\bar{\underline{a}}) + \underline{B}(\bar{\underline{a}})\underline{F}_y\underline{C} \end{array} \right] \tag{221}$$

and

$$\hat{\underline{X}}_{A_i}(k+1) = \underline{A}_{A_i}\hat{\underline{X}}_{A_i}(k) + \underline{B}_{A_i} v(k) \tag{222}$$

Then

$$m_{ik} = \sum_{j=1}^{N} \hat{\underline{X}}_{A_i}^T(j)\underline{hC}^T\underline{R}^{-1}\underline{Ch}^T\hat{\underline{X}}_{A_k}(j)$$

$$+ \sum_{j=1}^{N} \mathrm{tr}\left[\underline{h}^T \underline{P}_{A_{ik}}(j)\underline{hC}^T\underline{R}^{-1}\underline{C} \right] \tag{223}$$

where $\underline{h}^T$ is defined by Eq (51) and

$$\underline{P}_{A_{ik}}(j) = \underline{A}_{A_i}\underline{P}_{A_{ik}}(j-1)\underline{A}_{A_k}^T + \underline{D}_{A_i}\underline{RD}_{A_k}^T \tag{224}$$

$$\underline{P}_{A_{ik}}(0) = \underline{0} \tag{225}$$

Part of Eq (222) is not affected by different i's and need not be recomputed for each i.

The dimension of the information matrix is pxp so a scalar measure of $\underline{M}$ is required for the optimization criterion. Chapter I discussed some of the different scalar measures that could be used. These criteria include:

(1) maximize tr $\underline{M}$

(2) minimize tr $\underline{M}^{-1}$

(3) minimize $|\underline{M}^{-1}|$

(4) maximize tr ($\underline{SM}$) for $\underline{S}$ a given weighting matrix

It would be desirable to use the second or third criterion since they would minimize the sum of the parameter error variances or the volume in equiprobability contours respectively. Unfortunately, obtaining and working with $\underline{M}^{-1}$ is very difficult and mathematically impractical. For these reasons criteria 2 and 3 are usually not used.

The first criterion is the easiest to work with, and for this reason it is used in many cases. The problem with using this criterion, as stated by Zarrop and Goodwin (Ref 36), is that it may lead to an almost singular information matrix which has large diagonal terms. This would mean that the dispersion matrix, $\underline{M}^{-1}$, would also have large diagonal terms, that is, the lower bound on the error variances of the parameters would be large. This problem does not always occur, but is something that must be carefully checked.

The last criterion can be used in two ways. One method

is to preselect the weighting matrix based on known information and then hold it constant throughout the algorithm. The same problem that can occur with tr $\underline{M}$ could occur here with tr $\underline{SM}$, so it is not recommended. The second method is to select $\underline{S}$ such that by maximizing tr $\underline{SM}$, you are actually minimizing tr $\underline{M}^{-1}$ or $|\underline{M}^{-1}|$. An algorithm to select this $\underline{S}$ properly is presented by Gupta and Hall (Ref 11). Unfortunately the derivation of $\underline{S}$ to this date requires that the elements of $\underline{M}$, m_{ik}, only be affected by the input controls. This is not the case here since the feedback matrix affects m_{ik} and this matrix is being modified in the process of finding $\underline{F}_{yopt}$. Attempts of extending the derivation to this case were not fruitful, so this eliminates using only tr $\underline{SM}$ as a criterion. It is easy to see that each criterion has its advantages and disadvantages and a criterion that is a combination may be the best. For this reason the following ad hoc method is selected.

The criterion to maximize tr $\underline{M}$ is used with the addition that the value of $\underline{M}^{-1}$ will be computed at each iteration step. The overall objective is to decrease the volume in equiprobability contours, i.e., decrease $|\underline{M}^{-1}|$, so $|\underline{M}^{-1}|$ is also computed at each step. This means that as $\underline{F}_y$ is being iterated to maximize tr $\underline{M}$, $|\underline{M}^{-1}|$ is being monitored to be sure it is decreasing. If a point is reached where $|\underline{M}^{-1}|$ starts to increase, $\underline{F}_y$ is then held constant and the criterion to maximize tr $\underline{SM}$ is used where $\underline{S}$ is selected such

that $|\underline{M}^{-1}|$ is decreased. This procedure is written as an algorithm and is shown in Figure 12. The new feedback ma-

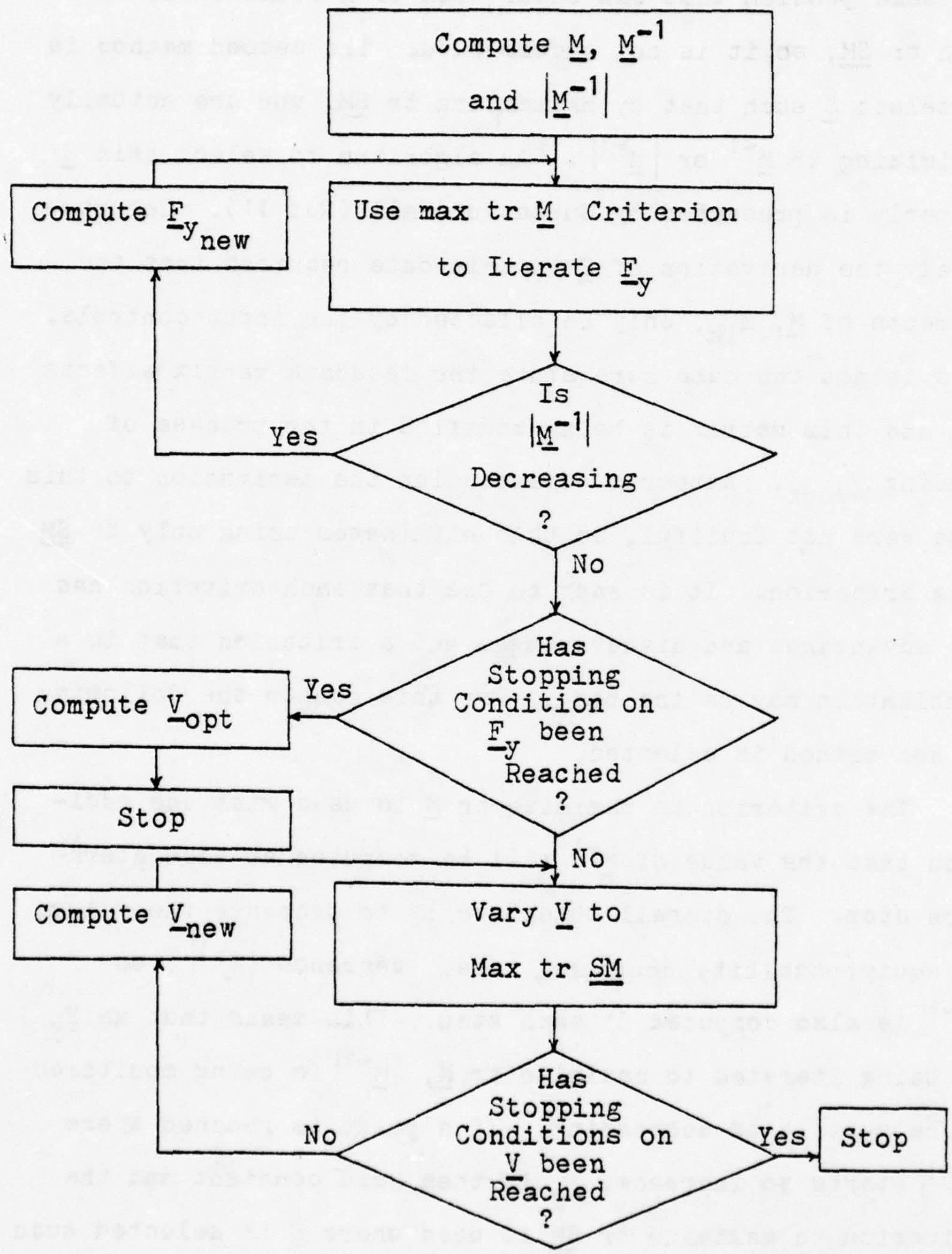

Figure 12. Multiple Parameter Algorithm

trix is computed using the algorithm of Figure 5, while the new optimal control sequence, $\underline{V}_{new}$, is computed using the method outlined in the weighted trace criterion section.

As mentioned, this is an ad hoc method and certainly does not guarantee that the global optimum values for $\underline{F}_y$ and $\underline{V}$ are found. For the nonlinear $\underline{M}$ derived in this research, none of the individual criteria given in this section can guarantee a global optimum value for $\underline{F}_y$ and $\underline{V}$. This method decreases $|\underline{M}^{-1}|$ while eliminating the computational burden required when working directly with $|\underline{M}^{-1}|$. In most cases the criterion of maximizing tr $\underline{SM}$ would probably not be required, which means the most tractable criterion would be used throughout the algorithm. This occurred in the examples that were tested, and the results appear later in this chapter. The next subsection will develop the max tr $\underline{M}$ and max tr $\underline{SM}$ criteria.

Trace M Criterion. The trace of $\underline{M}$ is written as

$$\mathrm{tr}\, \underline{M} = \sum_{i=1}^{p} \left[\underline{V}^T \underline{W}'_{N_i} \underline{V} + J'_i(\underline{F}_y) \right] \tag{226}$$

where

$$\underline{W}'_{N_i} = \begin{bmatrix} w'_{11_i} & w'_{12_i} & \cdots & w'_{1N_i} \\ w'_{21_i} & & & \\ \vdots & & & \\ w'_{N1_i} & & \cdots & w'_{NN_i} \end{bmatrix} \tag{227}$$

$$w'_{kl_i} = \sum_{j=\max(k,l)}^{N} \frac{\partial}{\partial \bar{a}_i}\left[\underline{A}_F^{j-k}(\bar{\underline{a}})\underline{B}(\bar{\underline{a}})\right]^T \underline{C}^T \underline{R}^{-1} \underline{C}$$

$$\cdot \frac{\partial}{\partial \bar{a}_i}\left[\underline{A}_F^{j-l}(\bar{\underline{a}})\underline{B}(\bar{\underline{a}})\right] \tag{228}$$

$$J'_i(\underline{F}_y) = \sum_{j=1}^{N} j\underline{F}_y \underline{R} \underline{F}_y{}^T \underline{B}_{A_i}^T (\underline{A}_{A_i}^T)^{N-j} \underline{hC}^T \underline{R}^{-1} \underline{Ch}^T \underline{A}_{A_i}^{N-j} \underline{B}_{A_i} \tag{229}$$

and N is the number of external control steps and p is the dimension of the unknown parameter vector. Equation (226) becomes:

$$\text{tr } \underline{M} = \underline{V}^T\left[\underline{W}'_{N_1} + \underline{W}'_{N_2} + \cdots + \underline{W}'_{N_p}\right]\underline{V}$$

$$+ J'_1(\underline{F}_y) + J'_2(\underline{F}_y) + \cdots + J'_p(\underline{F}_y)$$

$$= \underline{V}^T \underline{W}''_N \underline{V} + J''(\underline{F}_y) \tag{230}$$

where

$$\underline{W}''_N = \sum_{i=1}^{p} \underline{W}'_{N_i} \tag{231}$$

$$J''(\underline{F}_y) = \sum_{i=1}^{p} J'_i(\underline{F}_y) \tag{232}$$

Equation (230) is structured the same as Eq (84) so the procedure used to maximize M in Chapter II can be used here to maximize tr $\underline{M}$. That is, for $\underline{F}_y = \underline{F}_{yopt}$

$$\max \text{ tr } \underline{M} = \max_{\underline{V}} \underline{V}^T \underline{W}''_N \underline{V} + J''(\underline{F}_y)$$

$$= \lambda_{p_{max}} E + J''(\underline{F}_y) \tag{233}$$

where $\lambda_{p_{max}}$ is the maximum eigenvalue of $\underline{W}_N''$. The $\underline{F}_{yopt}$ is selected so that the maximum eigenvalue of $\underline{W}_N''$ is $\lambda_{p_{max}}$, and $\underline{V}$ is then the eigenvector of $\underline{W}_N''$ corresponding to $\lambda_{p_{max}}$ and scaled because of the energy constraint E. The computation has increased because of the complexity of $\underline{W}_N''$. The next subsection develops the criterion of maximizing tr $\underline{SM}$. As mentioned earlier, this criterion is required if $|\underline{M}^{-1}|$ starts to increase while tr $\underline{M}$ is being decreased.

Weighted Trace Criterion. The algorithm discussed in this subsection to compute $\underline{V}_{opt}$ to maximize tr $\underline{SM}$ is presented by Gupta and Hall (Ref 11). In their work feedback is not considered, but if feedback is used and held constant their approach can be applied. The value assigned to $\underline{F}_y$ would be the last value of $\underline{F}_y$ before $|\underline{M}^{-1}|$ started to increase.

The algorithm steps are:

(1) Let $\underline{V}_o$ be the value of $\underline{V}$ just before $|\underline{M}^{-1}|$ increased.

(2) Let $\underline{M}_o$ be the value of $\underline{M}$ before $|\underline{M}^{-1}|$ increased.

(3) Find a $\underline{V}_m$ with energy E that maximizes tr $(\underline{M}_o^{-1}\underline{M})$ and that yields $\underline{V}^T\underline{V}_m \geq 0$. $\underline{M}_o^{-1}$ is an initial weighting matrix and $\underline{M}$ is a function of only $\underline{V}$ since $\underline{F}_y$ has been determined. If tr $(\underline{M}^{-1}\underline{M}(\underline{V}_m))$ equals tr $(\underline{M}^{-1}\underline{M}(\underline{V}_o))$,

stop, since $\underline{V}_o$ is the optimal value for $\underline{V}$.

(4) Let $\underline{V}_k = \beta\underline{V}_o + \epsilon\underline{V}_m$ so the information matrix for $\underline{V}_k$ is

$$\underline{M}_k = \beta^2\underline{M}_o + \epsilon^2\underline{M}_m + 2\beta\epsilon\underline{M}_{om} \tag{234}$$

where $\underline{M}_m$ is the information matrix for input $\underline{V}_m$ and $\underline{M}_{om}$ is the "cross" information matrix for inputs $\underline{V}_o$ and $\underline{V}_m$ and is given as

$$\underline{M}_{om} = \underline{V}_o^T\underline{W}_N''\underline{V}_m + J''(\underline{F}_y) \tag{235}$$

The energy constraint for $\underline{V}_k$ is $\underline{V}_k^T\underline{V}_k \leq E$ which requires that $\beta^2\underline{V}_o^T\underline{V}_o$ and $\epsilon^2\underline{V}_m^T\underline{V}_m + 2\beta\epsilon\underline{V}_o^T\underline{V}_m \leq E$ or that

$$\beta^2E + \epsilon^2E + 2\beta\epsilon\underline{V}_o^T\underline{V}_m \leq E \tag{236}$$

thereby constraining the choice of β and ϵ.

(5) Now use Eqs (234) and (236) to find an ϵ between 0 and 1 which optimizes $\left|\underline{M}_k^{-1}\right|$ (ϵ is chosen arbitrarily to start, then iterated until the optimum is found; since (234) is quadratic, such an ϵ will eventually be found). This step requires the largest amount of computer time for this algorithm. Once ϵ is found, β can be found from Eq (236). Gupta and Hall show that if $\underline{V}_o$ is not optimal, then an improvement can always be made by using the computed β and ϵ in $\underline{V}_k = \beta\underline{V}_o + \epsilon\underline{V}_m$.

(6) Check for termination of the algorithm. Termination can occur when

a) the information matrix does not change substantially or

b) the value of ϵ is very close to zero.

If termination does not occur then $\underline{V}_k$ becomes $\underline{V}_o$, $\underline{M}_k$ becomes $\underline{M}_o$ and the process is repeated.

If the criterion is to minimize tr $\underline{M}^{-1}$ then in step 3 a $\underline{V}_m$ is found that maximizes tr $(\underline{M}_o^{-1}\underline{M}\underline{M}_o^{-1})$ and in step 5 an ϵ is found that minimizes tr $\underline{M}_k^{-1}$. The rest of the procedure remains the same. This completes the criteria derivation and it would be advantageous to present an example at this time.

Two Dimensional Unknown Parameter Example. The example problem is designed so that it is similar to the example in Chapter III. This should help the reader in following the problem. The estimator developed in Chapter III is modified slightly, as will be shown in this subsection, in order to handle the two dimensional case.

The equations that model the example system are

$$\underline{x}(j+1) = \begin{bmatrix} 0 & .1367\bar{a}_1^3 & 0 \\ -2\bar{a}_2 & 0 & \dfrac{-.1933}{\bar{a}_2^2} \\ 1-\bar{a}_1 & e^{1-2\bar{a}_2} & \dfrac{.75}{\bar{a}_2}e^{1-\bar{a}_1} \end{bmatrix} \underline{x}(j) +$$

$$+ \begin{bmatrix} 1-2\bar{a}_2 \\ 0 \\ e^{2(1-\bar{a}_1)} \end{bmatrix} u(j) \quad (237)$$

$$\underline{y}(j) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \underline{x}(j) + \bar{\underline{w}}(j) \quad (238)$$

The structure of $\underline{A}(\bar{\underline{a}})$ and $\underline{B}(\bar{\underline{a}})$ is chosen so that when the value of $\bar{\underline{a}}_o^T$ is set to $[1.0, .5]$, $\underline{A}(\bar{\underline{a}}_o)$ and $\underline{B}(\bar{\underline{a}}_o)$ are the same as in Chapter III.

The initial criterion that is used is to maximize tr $\underline{M}$. At the same time $|\underline{M}^{-1}|$ is monitored. If at anytime during the iterations to maximize tr $\underline{M}$, $|\underline{M}^{-1}|$ increases, the criterion to maximize tr $\underline{SM}$ is selected.

The constraint space on the eigenvalues is the same as in Chapter III, that is

$$|\lambda_{F_k}| < .9 \quad (189)$$

$$\mathrm{Re}\,\lambda_{F_k} \geq 0 \quad (190)$$

$$\mathrm{Imag}\,\lambda_{F_k} \leq \mathrm{Re}\,\lambda_{F_k} \quad (191)$$

The values of the covariance matrix for the noise, $\underline{R}$, is chosen to be $\underline{R} = 1.0\underline{I}$.

The algorithm of Figure 5 is used with the exception that because of the added dimension of $\bar{\underline{a}}$, the partial derivatives of a variable with respect to $\bar{\underline{a}}$ becomes twice as complex. For example, with one unknown parameter

$$\frac{\partial w'_{Nj}}{\partial \underline{F}_y^T} = \frac{\partial \underline{B}^T(\bar{\underline{a}})}{\partial \bar{a}_1} \underline{C}^T \underline{R}^{-1} \underline{C} \frac{\partial}{\partial \underline{F}_y^T} \left[\frac{\partial}{\partial \bar{a}_1} \left[\underline{A}_F^{N-j}(\bar{\underline{a}}) \underline{B}(\bar{\underline{a}}) \right] \right] \quad (239)$$

and with two unknown parameters it becomes:

$$\frac{\partial w''_{Nj}}{\partial \underline{F}_y^T} = \frac{\partial \underline{B}^T(\bar{\underline{a}})}{\partial \bar{a}_1} \underline{C}^T \underline{R}^{-1} \underline{C} \frac{\partial}{\partial \underline{F}_y^T} \left[\frac{\partial}{\partial \bar{a}_1} \left[\underline{A}_F^{N-j}(\bar{\underline{a}}) \underline{B}(\bar{\underline{a}}) \right] \right]$$

$$+ \frac{\partial \underline{B}^T(\bar{\underline{a}})}{\partial \bar{a}_2} \underline{C}^T \underline{R}^{-1} \underline{C} \frac{\partial}{\partial \underline{F}_y^T} \left[\frac{\partial}{\partial \bar{a}_2} \left[\underline{A}_F^{N-j}(\bar{\underline{a}}) \underline{B}(\bar{\underline{a}}) \right] \right] \quad (240)$$

During the iteration steps tr $\underline{M}$ increased from 577.4 to 12,806.6 and $|\underline{M}^{-1}|$ decreased from .00001358 to .00000016. At no time did $|\underline{M}^{-1}|$ increase, so the weighted trace criterion is never needed. The optimal values for $\underline{F}_y$ and $\underline{V}$ are

$$\underline{F}_{yopt} = \left[1.948, \ -.449 \right] \quad (241)$$

$$\underline{V}^T_{opt} = \left[.710, \ .557, \ .375, \ .200, \ .069, \ -.002 \right] \quad (242)$$

$\underline{F}_{yopt}$ is the same as for the two dimensional output example in Chapter III when $\underline{R} = 1.0\underline{I}$. This is not surprising since $\underline{A}(\bar{\underline{a}}_o)$ and $\underline{B}(\bar{\underline{a}}_o)$ have the same value for both examples. The partial derivatives given in Chapter III for the estimator are computed. Not only are partials required as shown in Chapter III, but now $\frac{\partial^2 \underline{A}(\bar{\underline{a}})}{\partial \bar{a}_2^2}$ and $\frac{\partial^2 \underline{A}(\bar{\underline{a}})}{\partial \bar{a}_1 \partial \bar{a}_2}$ are required as

well. These are needed because the matrix

$$\begin{bmatrix} \dfrac{\partial^2 L(\bar{\underline{a}})}{\partial \bar{a}_1^2} & \dfrac{\partial^2 L(\bar{\underline{a}})}{\partial \bar{a}_1 \partial \bar{a}_2} \\ \dfrac{\partial^2 L(\bar{a})}{\partial \bar{a}_1 \partial \bar{a}_2} & \dfrac{\partial^2 L(\bar{a})}{\partial \bar{a}_2^2} \end{bmatrix}$$

replaces $\dfrac{\partial^2 L(\bar{a})}{\partial \bar{a}^2}$ in the Newton-Raphson algorithm. The true value of the parameters is selected to be

$$\bar{\underline{a}}_T^T = \begin{bmatrix} 1.2, & .4 \end{bmatrix} \tag{243}$$

The modified estimator is used to perform the Monte Carlo analysis. Estimates are made with the feedback controls and with only open-loop controls, as in Chapter III. The number of steps in the control sequence is set at 6. Table VIII compares the results.

Table VIII. Average Estimate and Variance

For Parameter Vector

	Feedback	Open-loop
$\text{avg}\hat{\bar{a}}_1 = \hat{\bar{a}}_1'$	1.19512	1.22307
$\text{avg}\hat{\bar{a}}_2 = \hat{\bar{a}}_2'$	.40605	.38544
$\text{avg}(\bar{a}_{1T} - \hat{\bar{a}}_1)^2$	.001140	.016953
$\text{avg}(\bar{a}_{2T} - \hat{\bar{a}}_2)^2$	.000239	.000450

The optimal open-loop control sequence for this example is

$$\underline{V}^T = \left[.621,\ .612,\ .437,\ .214,\ .054,\ -.004\right] \tag{244}$$

It can be seen from Table VIII that both the estimate and the confidence in the estimate are improved for the feedback case.

Even though the weighted trace algorithm to minimize $|\underline{M}^{-1}|$ is not required, it is used to find the optimal open-loop controls to minimize $|\underline{M}^{-1}|$. After the optimal feedback matrix is computed, the weighted trace algorithm is applied, using this feedback, to determine if a better optimal external control sequence could be found. Table IX shows the results using both the max $\left[\text{tr } \underline{M}\right]$ and max $\left[\text{tr } \underline{SM}\right]$ criteria to compute the open-loop controls and Table X shows the results when the optimal feedback matrix is used.

Table IX. Optimal Open-loop Controls

	Criterion	
Value	tr $\underline{M}$	tr $\underline{SM}$
tr $\underline{M}$	577.4	576.8
$\lvert\underline{M}^{-1}\rvert$	.00001358	.00001355
t = 1	.621	.650
2	.612	.594
3	.437	.422
4	.214	.209
5	.054	.052
6	-.004	-.005

Table X. $\underline{V}_{opt}$ For Feedback Example

Value	Criterion: tr $\underline{M}$	Criterion: tr $\underline{SM}$
tr $\underline{M}$	12806.6	12779.5
$\lvert\underline{M}^{-1}\rvert$	.000000164	.000000163
t = 1	.710	.690
2	.557	.511
3	.375	.419
4	.200	.275
5	.069	.107
6	-.002	-.005

It can be seen that when the weighted trace criterion is used $\lvert\underline{M}^{-1}\rvert$ improved, but very slightly. The percentage improvement is well below 1% for both cases. It is obvious for this example that the decreased value of $\lvert\underline{M}^{-1}\rvert$ obtained when using tr $\underline{SM}$ does not justify the additional computation required to compute $\underline{S}$.

In summary, max tr $\underline{M}$ criterion is used because the weighted trace criterion can not handle the problem in which the elements of $\underline{M}$ are dependent on a variable other than the controls $\underline{V}$. The problem with the unweighted trace is that $\underline{M}$ may approach singularity and cause erroneous results, thereby necessitating the addition of a procedure to check for this condition. In the algorithm of Figure 12, this check is the computation of $\lvert\underline{M}^{-1}\rvert$. If this condition occurs,

further refinement is obtained by holding $\underline{F}_y$ constant and varying $\underline{V}$ only.

The modifications to the basic algorithm of Figure 5 to allow for multiple parameters are listed below. The other

Table XI. Changes to Blocks of Algorithm in Figure 5 to Compute $\underline{V}_{opt}$ and $\underline{F}_{yopt}$ For Multiple Parameters

Algorithm Block Number	Eq Number For Single Unknown Parameter	Corresponding Result For Multiple Unknown Parameters
2	83	228, 231
4	86, 87	87, 229, 232
5	-	Algorithm Fig. 12
15	93	Algorithm Fig. 12

blocks are only affected by the fact that the computation is increased because of the added unknown parameters. An example of this is shown in Eqs (239) and (240).

Multiple Input Controls

Chapter II addresses the single input control problem. There are many practical applications in which the dimension of the controls is more than one and this section modifies the previous results to handle this situation.

The system equations (14) and (15) become:

$$\underline{x}(k+1) = \underline{A}(\bar{a})\underline{x}(k) + \underline{B}(\bar{a})\underline{u}(k) \tag{245}$$

$$\underline{y}(k) = \underline{C}\underline{x}(k) + \underline{\bar{w}}(k) \tag{246}$$

where $\underline{B}(\bar{a})$ is an nxn time-invariant control matrix and $\underline{u}(k)$ is now an m-dimensional control vector. The feedback control is

$$\underline{u}(k) = \underline{F}_y \underline{y}(k) + \underline{v}(k) \tag{247}$$

where $\underline{F}_y$ is an mxr time-invariant feedback matrix and $\underline{v}(k)$ is an m-dimensional external control vector.

A relationship between the eigenvalues of the feedback system and the feedback matrix is required. This relationship is developed in Chapter II and by combining Eqs (153) and (155) is

$$\underline{F}_y = \underline{Z}(\underline{H} - \underline{L})\underline{T}^{-1}\underline{C}^T(\underline{C}\underline{C}^T)^{-1} \tag{248}$$

The derivation of $\underline{T}$, $\underline{L}$, and $\underline{Z}$ is developed in Appendix F and Eqs (140), (142), and (143) show how $\underline{H}$ is obtained. The consistency constraints are developed for the single input case in Chapter II, Eq (162). For the multiple input problem, the complexity of the constraint equations increases. Let

$$\underline{F}_s = \underline{F}_y \underline{C} \tag{249}$$

where $\underline{F}_s$ is mxn, $\underline{F}_y$ is mxr, and $\underline{C}$ is rxn. This can be written as

$$\begin{bmatrix} f_{s_{11}} & f_{s_{12}} & \cdots f_{s_{1n}} \\ f_{s_{21}} & & \\ \vdots & & \\ f_{s_{m1}} & \cdots & f_{s_{mn}} \end{bmatrix} = \begin{bmatrix} f_{y_{11}} & f_{y_{12}} & \cdots f_{y_{1r}} \\ f_{y_{21}} & & \\ \vdots & & \\ f_{y_{m1}} & \cdots & f_{y_{mr}} \end{bmatrix} \begin{bmatrix} \underline{c}_1 \cdots \underline{c}_n \end{bmatrix} \tag{250}$$

Again since $\underline{C}$ is of rank r, there are r independent columns and it is assumed that they are the first r columns. Then

$$\begin{bmatrix} f_{s_{11}} & f_{s_{12}} \cdots & f_{s_{1r}} \\ f_{s_{21}} & & \vdots \\ \vdots & & \\ f_{s_{m1}} & \cdots & f_{s_{mr}} \end{bmatrix} = \begin{bmatrix} f_{y_{11}} & f_{y_{12}} \cdots & f_{y_{1r}} \\ f_{y_{21}} & & \vdots \\ \vdots & & \\ f_{y_{m1}} & \cdots & f_{y_{mr}} \end{bmatrix} \underline{C}'^{-1} \tag{251}$$

where

$$\underline{C}' = \left[\underline{c}_1 \mid \underline{c}_2 \mid \cdots \mid \underline{c}_r\right] \tag{252}$$

Repeating Eq (161)

$$\underline{c}_i = \sum_{j=1}^{r} k_{ij}\underline{c}_j \tag{161}$$

where $i = r+1, r+2, \cdots, n$ and k_{ij} are coefficients. From Eq (250)

$$\underline{f}_{s \cdot i} = \underline{F}_y \underline{c}_i \tag{253}$$

where $\underline{f}_{s \cdot i}$ is the i-th column of F_s. Combining Eqs (161) and (253)

$$\underline{f}_{s \cdot i} = \underline{F}_y \sum_{j=1}^{r} k_{ij}\underline{c}_j$$

$$= \sum_{j=1}^{r} k_{ij}\underline{F}_y \underline{c}_j$$

$$= \sum_{j=1}^{r} k_{ij} \underline{f}_{s \cdot j} \tag{254}$$

Changing this vector equation to m scalar equations gives

$$\underline{f}_{s_{1i}} = \sum_{j=1}^{r} k_{ij} \underline{f}_{s_{1j}} \tag{255}$$

$$\underline{f}_{s_{2i}} = \sum_{j=1}^{r} k_{ij} \underline{f}_{s_{2j}} \tag{256}$$

$$\vdots$$

$$\underline{f}_{s_{mi}} = \sum_{j=1}^{r} k_{ij} \underline{f}_{s_{mj}} \tag{257}$$

It can be seen that since the dimension of the controls is now m, there are m times as many consistency constraint equality equations on the feedback system eigenvalues. Having the relationship between $\underline{\lambda}_F$ and $\underline{F}_y$, Eq (248), and the consistency constraints, Eqs (255) through (257), the changes to the algorithm in Figure 5 because of multiple inputs are developed.

Algorithm Modification. The criterion is the maximization of the information scalar because it is assumed that there is only one unknown parameter. If there were multiple parameters, the material developed in the previous section could be used. The information scalar M is given by

$$M = \sum_{j=1}^{N} \hat{\underline{X}}_A^T(j)\underline{hC}^T\underline{R}^{-1}\underline{Ch}^T\hat{\underline{X}}_A(j) + \sum_{j=1}^{N} tr\left[\underline{h}^T\underline{P}_A(j)\underline{hC}^T\underline{R}^{-1}\underline{C}\right] \quad (54)$$

where the relations corresponding to Eq (44) to Eq (60) are

$$\hat{\underline{X}}_A(j+1) = \underline{A}_A\hat{\underline{X}}_A(j) + \underline{B}_A\underline{v}(k) \quad (258)$$

$$\underline{B}_A^T = \left[\underline{B}^T(\bar{a}), \frac{\partial\, \underline{B}^T(\bar{a})}{\partial\, \bar{a}}\right]_{mx2n} \quad (259)$$

$$\underline{A}_A = \left[\begin{array}{c|c} \underline{A}(\bar{a}) + \underline{B}(\bar{a})\underline{F}_y\underline{C} & \underline{0} \\ \hline \frac{\partial\, \underline{A}(\bar{a})}{\partial\, \bar{a}} + \frac{\partial \underline{B}(\bar{a})}{\partial \bar{a}}\underline{F}_y\underline{C} & \underline{A}(\bar{a}) + \underline{B}(\bar{a})\underline{F}_y\underline{C} \end{array}\right]_{2nx2n} \quad (260)$$

and

$$\underline{P}_A(j) = \sum_{l=1}^{j} \underline{A}_A^{l-1}\underline{B}_A\underline{F}_y\underline{RF}_y{}^T\underline{B}_A^T(\underline{A}_A^T)^{l-1} \quad (261)$$

Equation (54) now becomes:

$$M = \sum_{j=1}^{N} \hat{\underline{X}}_A^T(j)\underline{hC}^T\underline{R}^{-1}\underline{Ch}^T\hat{\underline{X}}_A(j)$$
$$+ \sum_{j=1}^{N} j\,tr\left[\underline{h}^T\underline{A}_A^{N-j}\underline{B}_A\underline{F}_y\underline{RF}_y{}^T\underline{B}_A^T(\underline{A}_A^T)^{N-j}\underline{hC}^T\underline{R}^{-1}\underline{C}\right] \quad (262)$$

The trace operation cannot be eliminated as is done in the single input case. It is proven in Appendix A that the second term is non-negative.

In Chapter II the procedure was to express Eq (54) into the format

$$M = \underline{V}^T \underline{W}_N' \underline{V} + J(\underline{F}_y) \tag{84}$$

Equation (84) can then be maximized by using the given procedures. The following equations follow the work given by Eqs (54) through (84), but are modified because of the m-dimensional control vector. The increased dimension of $\underline{W}_N'$ and $\underline{V}$ clearly indicate the additional computation required to solve the multi-input problem.

Equation (65) becomes, for the multi-input case,

$$\hat{\underline{X}}_A(j) = \underline{A}_A^j \hat{\underline{X}}_A(0) + \underline{A}_A^{j-1} \underline{B}_A \underline{v}(0) + \underline{A}_A^{j-2} \underline{B}_A \underline{v}(1) + \cdots + \underline{A}_A^0 \underline{B}_A \underline{v}(j-1)$$

$$= \left[\underline{A}_A^j, \ \underline{A}_A^{j-1}\underline{B}_A, \cdots, \ \underline{B}_A\right]_{2n\mathrm{x}(2n+km)} \begin{bmatrix} \hat{\underline{X}}_A(0) \\ \underline{v}(0) \\ \cdot \\ \cdot \\ \cdot \\ \underline{v}(j-1) \end{bmatrix}_{(2n+km)\mathrm{x}1} \tag{263}$$

Letting

$$\underline{V}_X^T(j) = \left[\underline{x}^T(0), \ \underline{v}^T(0), \ \underline{v}^T(1), \cdots, \ \underline{v}^T(j-1)\right]_{1\mathrm{x}(n+km)} \tag{264}$$

then Eq (73) becomes:

$$\underline{E}^T(k) = \left[\begin{array}{c|c|c|c} \underline{A}_F^k(\bar{a}) & \underline{A}_F^{k-1}(\bar{a})\underline{B}(\bar{a}) & \cdots & \underline{B}(\bar{a}) \\ \hline \dfrac{\partial \, \underline{A}_F^k(\bar{a})}{\partial \, \bar{a}} & \dfrac{\partial}{\partial \, \bar{a}}\left[\underline{A}_F^{k-1}(\bar{a})\underline{B}(\bar{a})\right] & & \dfrac{\partial \underline{B}(\bar{a})}{\partial \, \bar{a}} \end{array}\right]_{2n\mathrm{x}(n+km)} \tag{265}$$

Eqs (75) and (76) remain the same and the analogy to Eq (78) is

$$K = \sum_{k=1}^{N} \underline{V}_X^T(k)\underline{W}(k)\underline{V}_X(k) \quad = \underline{V}_N^T\underline{W}_N\underline{V}_N \tag{266}$$

where

$$\underline{V}_N^T = \underline{V}_X^T(N)$$

$$= \left[\underline{x}^T(0),\ \underline{v}^T(0),\ \underline{v}^T(1), \cdots,\ \underline{v}^T(N-1)\right]_{1x(n+Nm)} \tag{267}$$

When the initial condition for the state is zero, Eq (266) becomes:

$$K = \underline{V}^T\underline{W}_N^{'}\underline{V} \tag{268}$$

where $\underline{V}^T = \left[\underline{v}^T(0),\ \underline{v}^T(1), \cdots,\ \underline{v}^T(N-1)\right]_{1xNm}$ and the dimension of $\underline{V}$ and $\underline{W}_N^{'}$ has been increased by a factor of m, the dimension of the input controls. This significantly increases the computation time required to compute $\underline{W}_N^{'}$ (now a square matrix of dimension Nm instead of N) and the maximum eigenvalue of $\underline{W}_N^{'}$ and its corresponding eigenvector. Equation (82) can still be used to compute $\underline{W}_N^{'}$ since the only change would be that $\underline{B}(\bar{a})$ is now a matrix instead of a vector. Using Eq (83) to compute the elements of $\underline{W}_N^{'}$ can be done; however, the result is no longer a scalar, but as shown, is an mxm matrix relationship.

$$\underline{w}_{kl}^{'} = \sum_{i=\max(k,l)}^{N} \frac{\partial}{\partial\bar{a}}\left[\underline{A}_F^{i-k}(\bar{a})\underline{B}(\bar{a})\right]^T\underline{C}^T\underline{R}^{-1}\underline{C}\ .$$

$$\cdot \frac{\partial}{\partial \bar{a}} \underline{A}_F^{i-1}(\bar{a}) \underline{B}(\bar{a}) \qquad (269)$$

Following the procedure of Chapter II, once the elements of $\underline{W}_N'$ are computed, the maximum eigenvalue and corresponding eigenvector can be evaluated. The second term of Eq (84) is

$$J(\underline{F}_y) = \sum_{j=1}^{N} j\,\mathrm{tr}\left[\underline{h}^T \underline{A}_A^{N-j} \underline{B}_A \underline{F}_y \underline{R} \underline{F}_y^T \underline{B}_A^T (\underline{A}_A^T)^{N-j} \underline{h} \underline{C}^T \underline{R}^{-1} \underline{C}\right] \qquad (270)$$

To compute the gradient, the $\frac{\partial \lambda_{max}(\underline{F}_y)}{\partial \underline{w}_{kl}'^T}$ and $\frac{\partial \underline{w}_{kl}'}{\partial \underline{F}_y^T}$ must be computed. Computing $\frac{\partial \lambda_{max}(\underline{F}_y)}{\partial \underline{w}_{kl}'^T}$ is not difficult since Eq (107) gives the partial of $\lambda_{max}(\underline{F}_y)$ with respect to each element of $\underline{W}_N'$ and by block partitioning the result becomes $\frac{\partial \lambda_{max}(\underline{F}_y)}{\partial \underline{w}_{kl}'^T}$. Finding the other partial is not as easy since it is taking the partial of a matrix with respect to another matrix. The procedure used in this research is to compute $\frac{\partial \underline{w}_{kl}'}{\partial f_{y_{ab}}}$ for each value of k, l, a, and b where

$$b = 1, 2, \cdots, r$$

$$k, l, a = 1, 2, \cdots, m$$

and $f_{y_{ab}}$ is an element of $\underline{F}_y$. Using this procedure

$$\frac{\partial \underline{w}_{kl}'}{\partial f_{y_{ab}}} = \sum_{i=\max(k,l)}^{N} \left[\frac{\partial}{\partial f_{y_{ab}}} \left[\frac{\partial}{\partial \bar{a}} \left[\underline{A}_F^{i-k}(\bar{a}) \underline{B}(\bar{a}) \right]^T \right] \underline{C}^T \underline{R}^{-1} \underline{C} \right.$$

$$\cdot \frac{\partial}{\partial \bar{a}} \left[\underline{A}_F^{i-l}(\bar{a}) \underline{B}(\bar{a}) \right] + \frac{\partial}{\partial \bar{a}} \left[\underline{A}_F^{i-k}(\bar{a}) \underline{B}(\bar{a}) \right] \underline{C}^T \underline{R}^{-1} \underline{C}$$

$$\left. \cdot \frac{\partial}{\partial f_{y_{ab}}} \left[\frac{\partial}{\partial \bar{a}} \left[\underline{A}_F^{i-l}(\bar{a}) \underline{B}(\bar{a}) \right] \right] \right] \qquad (271)$$

where the iterative equation

$$\frac{\partial}{\partial f_{y_{ab}}} \left[\frac{\partial}{\partial \bar{a}} \left[\underline{A}_F^{m}(\bar{a}) \underline{B}(\bar{a}) \right] \right] = \underline{A}_F(\bar{a}) \frac{\partial}{\partial f_{y_{ab}}} \left[\frac{\partial}{\partial \bar{a}} \left[\underline{A}_F^{m-1}(\bar{a}) \underline{B}(\bar{a}) \right] \right]$$

$$+ \frac{\partial}{\partial f_{y_{ab}}} \left[\underline{A}_F(\bar{a}) \right] \frac{\partial}{\partial \bar{a}} \left[\underline{A}_F^{m-1}(\bar{a}) \underline{B}(\bar{a}) \right] + \frac{\partial}{\partial f_{y_{ab}}} \left[\frac{\partial}{\partial \bar{a}} \underline{A}_F(\bar{a}) \right]$$

$$\cdot \left[\underline{A}_F^{m-1}(\bar{a}) \underline{B}(\bar{a}) \right] + \frac{\partial \underline{A}_F(\bar{a})}{\partial \bar{a}} \frac{\partial}{\partial f_{y_{ab}}} \left[\underline{A}_F^{m-1}(\bar{a}) \underline{B}(\bar{a}) \right] \qquad (272)$$

and

$$\frac{\partial}{\partial f_{y_{ab}}} \left[\frac{\partial}{\partial \bar{a}} \left[\underline{A}_F^{0}(\bar{a}) \underline{B}(\bar{a}) \right] \right] = \underline{0} \qquad (273)$$

can be used. Equations (271) through (273) are used to compute $\frac{\partial \underline{w}_{kl}'}{\partial f_{y_{ab}}}$ and by performing this for each element of $\underline{F}_y$, $\frac{\partial \lambda_{max}(\underline{F}_y)}{\partial \underline{F}_y^T}$ can be computed.

It would be useful at this time to consider an example to demonstrate how $\frac{\partial \lambda_{max}(\underline{F}_y)}{\partial \underline{F}_y^T}$ is computed. Let m = 2, k = l = 1, so

$$\frac{\partial \lambda_{max}(\underline{F}_y)}{\partial \underline{w}_{11}^{'T}} = \begin{bmatrix} \frac{\partial \lambda_{max}(\underline{F}_y)}{\partial (w_{11}^{'})_{11}} & \frac{\partial \lambda_{max}(\underline{F}_y)}{\partial (w_{11}^{'})_{12}} \\ \frac{\partial \lambda_{max}(\underline{F}_y)}{\partial (w_{11}^{'})_{21}} & \frac{\partial \lambda_{max}(\underline{F}_y)}{\partial (w_{11}^{'})_{22}} \end{bmatrix} \tag{274}$$

If a, b = 1 in Eq (271) then

$$\frac{\partial \underline{w}_{11}^{'}}{\partial f_{y_{11}}} = \begin{bmatrix} \frac{\partial (w_{11}^{'})_{11}}{\partial f_{y_{11}}} & \frac{\partial (w_{11}^{'})_{12}}{\partial f_{y_{11}}} \\ \frac{\partial (w_{11}^{'})_{21}}{\partial f_{y_{11}}} & \frac{\partial (w_{11}^{'})_{22}}{\partial f_{y_{11}}} \end{bmatrix} \tag{275}$$

The overall result that is desired is to find the value of $\frac{\partial \lambda_{max}(\underline{F}_y)}{\partial f_{y_{11}}}$ which is

$$\frac{\partial \lambda_{max}(\underline{F}_y)}{\partial f_{y_{11}}} = \sum_{i=1}^{2} \sum_{j=1}^{2} \frac{\partial \lambda_{max}(\underline{F}_y)}{\partial (w_{11}^{'})_{ij}} \frac{\partial (w_{11}^{'})_{ij}}{\partial f_{y_{11}}}$$

$$+ \text{ summation for } (k,l) = (1,2), (2,1), (2,2) \tag{276}$$

In general, when the values for the partials in Eqs (274)

and (275) are known, the following equation is needed to obtain one element of $\frac{\partial \lambda_{max}(\underline{F}_y)}{\partial \underline{F}_y^T}$:

$$\frac{\partial \lambda_{max}(\underline{F}_y)}{\partial f_{y_{ab}}} = \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{N} \sum_{l=1}^{N} \frac{\partial \lambda_{max}(\underline{F}_y)}{\partial (w'_{kl})_{ij}} \frac{\partial (w'_{kl})_{ij}}{\partial f_{y_{ab}}} \quad (277)$$

By performing this for each value of a and b, $\frac{\partial \lambda_{max}(\underline{F}_y)}{\partial \underline{F}_y^T}$ can be computed. This appears to be a time consuming operation, but most of the computation is iterative and can be done very quickly on the computer.

It can be seen from Eq (101) that $\frac{\partial J(\underline{F}_y)}{\partial \underline{F}_y^T}$ is required. Equation (270) gives $J(\underline{F}_y)$ for the multi-input case and, as mentioned earlier, the trace operation cannot be eliminated as done in the single input case. Equation (270) becomes:

$$J(\underline{F}_y) = \sum_{j=1}^{N} j\mathrm{tr}\left[\frac{\partial}{\partial \bar{a}}\left[\underline{A}_F^{N-j}(\bar{a})\underline{B}(\bar{a})\right] \underline{F}_y \underline{R} \underline{F}_y^T \cdot \frac{\partial}{\partial \bar{a}}\left[\underline{A}_F^{N-j}(\bar{a})\underline{B}(\bar{a})\right]^T \underline{C}^T \underline{R}^{-1} \underline{C}\right] \quad (278)$$

If an attempt were made to find $\frac{\partial J(\underline{F}_y)}{\partial \underline{F}_y^T}$ directly, eventually the partial of a matrix with respect to another matrix would be required. To avoid this, the partial is evaluated with respect to each element of $\underline{F}_y$.

$$\frac{\partial J(\underline{F}_y)}{\partial f_{y_{ab}}} = \sum_{j=1}^{N} j\,tr\left[\frac{\partial}{\partial f_{y_{ab}}} \underline{X}\underline{F}_y \underline{R}\underline{F}_y{}^T \underline{X}^T \underline{C}^T \underline{R}^{-1} \underline{C}\right.$$

$$\left. + \underline{X}\frac{\partial}{\partial f_{y_{ab}}}\left[\underline{F}_y \underline{R}\underline{F}_y{}^T\right] \underline{X}^T \underline{C}^T \underline{R}^{-1} \underline{C} + \underline{X}\underline{F}_y \underline{R}\underline{F}_y{}^T \frac{\partial}{\partial f_{y_{ab}}} \underline{X}^T \underline{C}^T \underline{R}^{-1} \underline{C}\right] \quad (279)$$

where

$$\underline{X} = \frac{\partial}{\partial \bar{a}}\left[\underline{A}_F^{N-j}(\bar{a})\underline{B}(\bar{a})\right] \quad (280)$$

Equations (272) and (273) show how to compute $\frac{\partial}{\partial f_{y_{ab}}}\underline{X}$. Also

$$\frac{\partial}{\partial f_{y_{ab}}} \underline{F}_y \underline{R}\underline{F}_y{}^T = \underline{Y}\underline{R}\underline{F}_y{}^T + \underline{F}_y \underline{R}\underline{Y}^T \quad (281)$$

where $\underline{Y}$ is an mxr matrix consisting of all zeros except for a one in the a,b position. Performing Eq (279) for all elements of $\underline{F}_y$ gives $\frac{\partial J(\underline{F}_y)}{\partial \underline{F}_y{}^T}$. It is now possible to compute $\frac{\partial M_m(\underline{F}_y)}{\partial \underline{F}_y{}^T}$ from Eq (101).

To complete the algorithm $\frac{\partial \underline{F}_y}{\partial \underline{\lambda}_F{}^T}$ is required. The procedure to use is again to compute first $\frac{\partial \underline{F}_y}{\partial \lambda_{F_k}}$ for each value of k. From Eq (248)

$$\frac{\partial \underline{F}_y}{\partial \lambda_{F_k}} = \underline{Z}\frac{\partial \underline{H}}{\partial \lambda_{F_k}} \underline{T}^{-1} \underline{C}^T (\underline{C}\underline{C}^T)^{-1} \quad (282)$$

where Eqs (140) and (166) can be used to compute $\frac{\partial \underline{H}}{\partial \lambda_{F_k}}$. Repeating Eq (163)

$$\left.\frac{\partial M_m(\underline{F}_y)}{\partial \underline{\lambda}_F^T}\right|_{\underline{\lambda}_F = \underline{\lambda}_{Fc}} = \left.\frac{\partial M_m(\underline{F}_y)}{\partial \underline{F}_y^T}\right|_{\underline{F}_y = \underline{F}_{yc}} \left.\frac{\partial \underline{F}_y^{T*}}{\partial \underline{\lambda}_F^T}\right|_{\underline{\lambda}_F = \underline{\lambda}_{Fc}} \qquad (163)$$

Knowing $\frac{\partial M_m(\underline{F}_y)}{\partial \underline{F}_y^T}$ and $\frac{\partial \underline{F}_y^*}{\partial \lambda_{F_k}}$, this is replaced by

$$\frac{\partial M_m(\underline{F}_y)}{\partial \lambda_{F_k}} = \sum_{a=1}^{m} \sum_{b=1}^{r} \frac{\partial M_m(\underline{F}_y)}{\partial f_{y_{ab}}} \frac{\partial f^*_{y_{ab}}}{\partial \lambda_{F_k}} \qquad (283)$$

Performing Eq (283) for each value of k then gives $\frac{\partial M_m(\underline{F}_y)}{\partial \underline{\lambda}_F^T}$.

This is the required gradient to maximize $M_m(\underline{F}_y)$ and completes the modifications needed to the algorithm equations of Figure 5. Table XII lists the differences between the single and multiple control algorithm.

The main difference between the single input case and the multi-input case is the additional computational burden resulting from multiple inputs. The optimality criterion and general approach to solving the problem is exactly the same for both cases. It can be seen that if a problem has multiple unknown parameters as well as multiple inputs, the computational requirements are enormous except for the simplest cases.

Table XII. Changes to Blocks of Algorithm in Figure 5 to Compute $\underline{V}_{opt}$ and $\underline{E}_{yopt}$ For Multiple Inputs

Algorithm Block Number	Eq Number For Single Input Control	Corresponding Result For Multiple Input Controls
2	83	269
4	86, 87	87, 270
7	107, 113	107, 271
8	101, 120	101, 279
9	165	282
10	163	283
12	153, 155	248

V Conclusions And Suggestions For Further Research

In this Chapter the major objectives of the research as well as the contributions are discussed. Also discussed are areas for future research, all of which are extensions to the research presented in this dissertation.

Objectives And Contributions

The objective of this research is to show that the estimation of unknown parameters in the plant and control matrices of a linear system can be enhanced by the addition of a constant gain output feedback control. To attain this objective, an algorithm is developed in Chapter II that optimally selected the feedback vector and external control sequence for a single unknown parameter, single input control problem. An energy constraint is placed on the external control sequence and the feedback eigenvalues are confined to a constraint space. This space is selected so as to fulfill stability as well as response criteria for the system. It is also shown that for the case in which the number of outputs is less than the number of plant states, an additional constraint is placed on the feedback vector. This additional "consistency" constraint is required in order that the position of the feedback eigenvalues could be reached by feeding back the outputs.

The optimization criteria that are chosen are maximiz-

ing the trace and weighted trace of the Fisher information matrix. These criteria were selected because they increase the information about the unknown parameters in the output. By properly selecting the weighting matrix, maximizing tr $\underline{SM}$ is equivalent to minimizing the determinant of the inverse of the information matrix. Minimizing the determinant reduces the volume within contours of equal probability.

A closed form solution to the optimization problem does not exist, so an iterative technique must be used. The technique selected is a first order gradient approach, and it is used to find the values of $\underline{F}_{yopt}$ and $\underline{V}_{opt}$ because of the guarantee of convergence to a local solution and the simplicity over a higher order solution. To account for the problem of moving the eigenvalues in the direction that maximized the gradient while remaining in the constraint space, the gradient projection method is applied.

In Chapter III a single unknown parameter example is used to demonstrate the use of the derived algorithm. The unknown parameter appears nonlinearly throughout the plant and control matrices. A maximum likelihood estimator is developed and used to estimate the unknown parameter. Simulated system outputs are corrupted by noise from a Gaussian noise generator. The estimate error sample means and standard deviations are plotted for different noise levels and control steps. The plots confirmed that the addition of feedback controls result in better estimates, especially for large $\underline{R}$.

In Chapter IV the multiple input and multiple parameter cases are examined. Both result in substantially increased complexity of the problem, and the computation time required to solve the problem could become a significant limiting factor.

It is difficult to select a good scalar criterion for the multiple parameter cases. The more appealing criterion, because of the simpler computations required, could result in nonidentifiable solutions. The criterion that would produce an estimate with a minimum error covariance usually becomes mathematically intractable. As a compromise, a combined criterion was selected. This ad hoc method consisted of using the maximization of the trace of the information matrix as the criterion as long as the determinant of the inverse of the information matrix decreased. If this did not occur, then a weighted trace criterion was used, where the weighting matrix is selected such that the determinant did decrease. This required that the feedback matrix remain constant.

In the multiple input case the complexity increased rapidly as the dimension of the input controls increased. Only cases in which the dimension of the input controls is small would be practical to solve.

Lopez-Toledo (Ref 17) addressed the feedback control problem, but his model was linear in the parameters. He also did not address the problem of confining the feedback

system eigenvalues. A contribution of this research is the solution of feedback controls in which the parameters are nonlinear in the model. Another contribution is that the solution allows constraints to be placed on the eigenvalues. The computational method developed in this research reduced the computer time required to solve the problem. By setting the problem up in a desired format

$$M = \underline{V}^T \underline{W}_N' \underline{V} + J(\underline{F}_y) \tag{84}$$

maximizing M can be accomplished by first maximizing

$$\lambda_{max}(\underline{F}_y)E + J(\underline{F}_y)$$

where $\lambda_{max}(\underline{F}_y)$ is the maximum eigenvalue of $\underline{W}_N'$ and E is the energy constraint on $\underline{V}$, with respect to $\underline{F}_y$ only. Once this is accomplished then the controls vector $\underline{V}$ is the scaled eigenvector corresponding to the maximum value of $\lambda_{max}(\underline{F}_y)$. This method allows the feedback matrix to be solved separately from the external open-loop controls.

The method and results presented in this research should encourage future research in this area. Some of the areas that need further research are discussed in the next section.

Extension For Further Research

The most important area that needs further research is the case in which the initial conditions of the states are neither zero nor free to be chosen. Since the research re-

quired zero initial states or freedom to select them, the system would have to be initialized to these conditions which in some situations may be neither practical nor possible.

One example may be where the system is initialized properly and a certain time later the outputs and applied controls are used to estimate the parameters. These estimates could then be used as a better nominal parameter value for future runs, but the system is probably not at the zero state. Future runs could not occur until the system was reinitialized, which may cause a significant delay.

Another area for research would be to investigate modern computational techniques to decrease the computer time required to obtain a solution. It was found that a significant amount of time was required even for the simple three state problem. In order to apply the method to practical problems, further research should look into considering the computational requirements. A rewarding output of this new research may be the application of this technique to real-time problems.

Another area would be to investigate the possibility of using this research in the case where there is small process noise. Chapter I mentioned the additional complexity that may result when incorporating process noise into the system; however, further research may show that for small values of this noise, approximations may be used along with the procedures developed in this research.

One area that was briefly considered was the case in which the value of the measurement noise used by the estimator was different from the real value. For small mistuned $\underline{R}$'s, the estimates were not affected substantially, but more time was required by the estimator. More work should be spent in studing the effects of mistuned $\underline{R}$'s since this robustness issue partially answers the question of how well this method will work on real, rather than simulated, data.

It was noticed that for the case where $\underline{C}^{-1}$ exists, all the eigenvalues moved to the boundary of the constraint space and tried to cause the system to become unstable. This was not surprising since maximal excitation of modes enhances identification; however, attempts at proving that this in fact is always the case were not fruitful. Further study in this area may be very beneficial.

The final suggested area for further research is to determine if there is an optimal set of columns of $\underline{C}$ for computing the consistency constraints. There may be a set of columns that gives constraints that produces a better input control. Even if all sets yield the same controls, one set may minimize the computation needed to compute $\underline{F}_{yopt}$ and $\underline{V}_{opt}$. This would be worth investigating further.

The purpose of the dissertation was to develop an optimal feedback control that will enhance the identification of unknown parameters. This was accomplished and hopefully it will initiate an interest in applying this to future problems.

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Appendix A

$$\text{Proof That } \sum_{j=1}^{N} \operatorname{tr}\left[\underline{h}^T \underline{P}_A(j)\underline{h}\underline{C}^T\underline{R}^{-1}\underline{C}\right] \geq 0$$

Let

$$\underline{G}(j) = \underline{h}^T \underline{P}_A(j)\underline{h} \tag{A-1}$$

$$\underline{E} = \underline{C}^T\underline{R}^{-1}\underline{C} \tag{A-2}$$

Since $\underline{P}_A(j)$ and $\underline{R}^{-1}$ are real, symmetric, positive semidefinite matrices, $\underline{G}$ and $\underline{E}$ are also real, symmetric, positive semidefinite matrices. Let

$$\gamma_j = \operatorname{tr} \underline{G}(j)\underline{E} \tag{A-3}$$

Since $\underline{G}(j)$ is real and symmetric, there exists an orthogonal transformation matrix $\underline{P}$ such that

$$\underline{P}^T\underline{G}(j)\underline{P} = \underline{D}(j) \tag{A-4}$$

where $\underline{D}(j)$ is a diagonal matrix. Now

$$\begin{aligned} \underline{P}^T\underline{G}(j)\underline{E}\underline{P} &= \underline{P}^T\underline{G}(j)\underline{P}\underline{P}^T\underline{E}\underline{P} \\ &= \underline{D}(j)\underline{P}^T\underline{E}\underline{P} \\ &= \underline{D}(j)\underline{J} \end{aligned} \tag{A-5}$$

where $\underline{J} = \underline{P}^T\underline{E}\underline{P}$.

For all $\underline{x}$, where $\underline{x}$ is a vector

$$\underline{x}^T\underline{J}\underline{x} = \underline{x}^T(\underline{P}^T\underline{E}\underline{P})\underline{x}$$

$$= (\underline{Px})^T \underline{E}(\underline{Px})$$

$$\geq 0 \quad \text{since } \underline{E} \geq 0 \tag{A-6}$$

which implies $J \geq 0$. Since the trace is invariant under a similar transformation, and since $\underline{P}$ is orthogonal,

$$\begin{aligned} \text{tr } \underline{G}(j)\underline{E} &= \text{tr } \underline{P}^{-1}\underline{G}(j)\underline{EP} \\ &= \text{tr } \underline{P}^T\underline{G}(j)\underline{EP} \\ &= \text{tr } \underline{D}(j)\underline{J} \\ &= \sum_{i=1}^{n} d_{ii} j_{ii} \end{aligned} \tag{A-7}$$

$\underline{D}(j)$, $\underline{J} \geq 0$ implies d_{ii}, $j_{ii} \geq 0$ since if $\underline{x}^T = \begin{bmatrix} \overset{1}{0}, \cdots, \overset{i-1}{0}, \overset{i}{1}, \overset{i+1}{0}, \cdots, \overset{n}{0} \end{bmatrix}$ then

$$0 \leq \underline{x}^T \underline{Jx} = j_{ii} \tag{A-8}$$

$$0 \leq \underline{x}^T \underline{D}(j) \underline{x} = d_{ii} \tag{A-9}$$

So

$$\text{tr } \underline{G}(j)\underline{E} = \sum_{i=1}^{n} d_{ii} j_{ii} \geq 0 \tag{A-10}$$

Therefore,

$$\gamma_j \geq 0 \tag{A-11}$$

so

$$\sum_{j=1}^{N} \text{tr}\left[\underline{h}^T \underline{P}_A(j) \underline{hC}^T \underline{R}^{-1} \underline{C}\right] = \sum_{j=1}^{N} \gamma_j \geq 0 \tag{A-12}$$

This holds true for any feedback matrix $\underline{F}_y$.

Appendix B

Definition of a Vector Derivative

Let the scalar $x(\underline{y})$ be a function of the n dimensional vector $\underline{y}$. By convention, the definition of the derivative is:

$$\frac{\partial x(\underline{y})}{\partial \underline{y}} = \left[\frac{\partial x(\underline{y})}{\partial y_1}, \frac{\partial x(\underline{y})}{\partial y_2}, \cdots, \frac{\partial x(\underline{y})}{\partial y_n}\right] \tag{B-1}$$

This is sometimes called a gradient and physically it is finding the individual gradients of $x(\underline{y})$ with respect to a component of $\underline{y}$ with all other components held constant.

In the n+1 space Z, where

$$Z = \left[z: z = \left[x, y_1, y_2, \cdots, y_n\right]\right] \tag{B-2}$$

Eq (B-1) represents a hyperplane that is tangent to the surface of $x(\underline{y})$ at the point where the derivative was evaluated.

Appendix C

Computation of $\underline{A}_A^k$

If

$$\underline{A}_A = \left[\begin{array}{c|c} \underline{A}_F(\bar{a}) & \underline{0} \\ \hline \dfrac{\partial \underline{A}_F(\bar{a})}{\partial \bar{a}} & \underline{A}_F(\bar{a}) \end{array}\right] \tag{C-1}$$

then

$$\underline{A}_A^k = \left[\begin{array}{c|c} \underline{A}_F^k(\bar{a}) & \underline{0} \\ \hline \dfrac{\partial}{\partial \bar{a}}\underline{A}_F^k(a) & \underline{A}_F^k(\bar{a}) \end{array}\right] \tag{C-2}$$

Proof:

The proof uses induction. It is obvious that the statement is true for $k = 1$. Assume it is true for $k = j$, then for $k = j+1$

$$\begin{aligned} \underline{A}_A^{j+1} &= \underline{A}_A^j \underline{A}_A \\ &= \left[\begin{array}{c|c} \underline{A}_F^j(\bar{a}) & \underline{0} \\ \hline \dfrac{\partial\, \underline{A}_F^j(\bar{a})}{\partial \bar{a}} & \underline{A}_F^j(\bar{a}) \end{array}\right] \left[\begin{array}{c|c} \underline{A}_F(\bar{a}) & \underline{0} \\ \hline \dfrac{\partial \underline{A}_F(\bar{a})}{\partial \bar{a}} & \underline{A}_F(\bar{a}) \end{array}\right] \\ &= \left[\begin{array}{c|c} \underline{A}_F^{j+1}(\bar{a}) & \underline{0} \\ \hline \left[\dfrac{\partial}{\partial \bar{a}}\underline{A}_F^j(\bar{a})\right]\underline{A}_F(\bar{a}) + \underline{A}_F^j(\bar{a})\left[\dfrac{\partial}{\partial \bar{a}}\underline{A}_F(\bar{a})\right] & \underline{A}_F^{j+1}(\bar{a}) \end{array}\right] \end{aligned} \tag{C-3}$$

Since

$$\frac{\partial}{\partial \bar{a}} \underline{A}_F^{j+1}(\bar{a}) = \frac{\partial}{\partial \bar{a}}\left[\underline{A}_F^{j}(\bar{a})\underline{A}_F(\bar{a})\right]$$

$$= \left[\frac{\partial}{\partial \bar{a}} \underline{A}_F^{j}(\bar{a})\right] \underline{A}_F(\bar{a}) + \underline{A}_F^{j}(\bar{a}) \frac{\partial \underline{A}_F(\bar{a})}{\partial \bar{a}} \tag{C-4}$$

then

$$\underline{A}_A^{j+1} = \left[\begin{array}{c|c} \underline{A}_F^{j+1}(\bar{a}) & \underline{0} \\ \hline \dfrac{\partial \underline{A}_F^{j+1}(\bar{a})}{\partial \bar{a}} & \underline{A}_F^{j+1}(\bar{a}) \end{array}\right] \tag{C-5}$$

which implies that it is true for the case $k = j+1$, therefore, it is true for all k.

QED

Appendix D

Proof That all Available Control Energy is Utilized

It is seen from Eqs (22) and (83) that $\underline{W}_N'$ is a real, symmetric, positive semidefinite matrix. Let $\underline{F}_y$ be constant and let

$$\max M = M_1 = \underline{V}_1^T \underline{W}_N' \underline{V}_1 + J(\underline{F}_y) \tag{D-1}$$

Assume $0 < \underline{V}_1^T \underline{V}_1 < E$ and let $\underline{V}_2 = b\underline{V}_1$ where b is such that

$$\underline{V}_2^T \underline{V}_2 = E \tag{D-2}$$

This implies that

$$b^2 \underline{V}_1^T \underline{V}_1 = E \tag{D-3}$$

and

$$b^2 > 1 \tag{D-4}$$

Now

$$M_2 = \underline{V}_2^T \underline{W}_N' \underline{V}_2 + J(\underline{F}_y) \leq M_1 \tag{D-5}$$

From the fact that $M_1 = \max M$.

$$\begin{aligned} M_2 &= \underline{V}_2^T \underline{W}_N' \underline{V}_2 + J(\underline{F}_y) \\ &= b^2 \underline{V}_1^T \underline{W}_N' \underline{V}_1 + J(\underline{F}_y) \\ &> \underline{V}_1^T \underline{W}_N' \underline{V}_1 + J(\underline{F}_y) \\ &> M_1 \end{aligned} \tag{D-6}$$

This contradicts Eq (D-5) so the assumption that $0 < \underline{V}_1^T \underline{V}_1 < E$ is invalid.

Now assume $\underline{V}_1^T \underline{V}_1 = 0$ which implies that $\underline{V}_1 = 0$. Eq (D-1) now becomes:

$$\max M = M_1 = J(\underline{F}_y) \tag{D-7}$$

Let $\underline{V}_2^T \underline{V}_2 = E$ and since $\underline{W}_N'$ is positive semidefinite there exists a $\underline{V}_2$ such that

$$\begin{aligned} M_2 &= \underline{V}_2^T \underline{W}_N' \underline{V}_2 + J(\underline{F}_y) \\ &> J(\underline{F}_y) \\ &> M_1 \end{aligned} \tag{D-8}$$

This also contradicts Eq (D-5) so the assumption that $\underline{V}_1^T \underline{V}_1 = 0$ is invalid. Therefore,

$$\underline{V}_1^T \underline{V}_1 = E \tag{D-9}$$

QED

Appendix E

Simplified Vector Partial Derivatives

Let

$\underline{A}$ be an nxn matrix

$\underline{C}$ be an rxn matrix

$\underline{B}$, $\underline{D}$ be n-dimensional vectors

$\underline{F}$ be an r-dimensional vector

Then

$$\frac{\partial}{\partial \underline{F}}\left[\underline{A} + \underline{BF}^T\underline{C}\right]\underline{D} = \underline{BD}^T\underline{C}^T \tag{E-1}$$

and

$$\frac{\partial}{\partial \underline{F}}\left[\underline{A} + \underline{BF}^T\underline{C}\right]^k\underline{D} = \sum_{l=1}^{k}\left[\underline{A} + \underline{BF}^T\underline{C}\right]^{l-1}\underline{BD}^T\left[\underline{A} + \underline{BF}^T\underline{C}\right]^{k-l^T}\underline{C}^T \tag{E-2}$$

Proof:

$$\frac{\partial}{\partial \underline{F}}\left[\underline{A} + \underline{BF}^T\underline{C}\right]\underline{D} = \frac{\partial}{\partial \underline{F}}\underline{AD} + \frac{\partial}{\partial \underline{F}}(\underline{BF}^T\underline{CD})$$

$$= \frac{\partial}{\partial \underline{F}}\underline{BF}^T\underline{CD}$$

$$= \underline{B}\frac{\partial}{\partial \underline{F}}(\underline{F}^T\underline{CD}) \tag{E-3}$$

because $\underline{F}^T\underline{CD}$ is a scalar.

$$\frac{\partial}{\partial \underline{F}}\left[\underline{A} + \underline{BF}^T\underline{C}\right]\underline{D} = \underline{B}\frac{\partial}{\partial \underline{F}}\left[\left[f_1, f_2, \cdots, f_r\right]\left[\underline{c}_{k1}, \cdots, \underline{c}_{kn}\right]\underline{D}\right]$$

$$= \underline{B}\underline{c}_{kl}\underline{D} = \underline{B}\underline{D}^T\underline{c}_{kl}^T \tag{E-4}$$

where $\underline{c}_{kl}$ is the l-th row of $\underline{C}$.

Therefore,

$$\frac{\partial}{\partial \underline{F}}\left[\underline{F}^T\underline{C}\underline{D}\right] = \left[\underline{D}^T\underline{c}_{k1}^T \,\middle|\, \underline{D}^T\underline{c}_{k2}^T \,\middle|\, \cdots \,\middle|\, \underline{D}^T\underline{c}_{kr}^T\right]$$

$$= \underline{D}^T\left[\underline{c}_{k1}^T \,\middle|\, \underline{c}_{k2}^T \,\middle|\, \cdots \,\middle|\, \underline{c}_{kr}^T\right]$$

$$= \underline{D}^T\underline{C}^T \tag{E-5}$$

so

$$\frac{\partial}{\partial \underline{F}}\left[\underline{A} + \underline{B}\underline{F}^T\underline{C}\right]\underline{D} = \underline{B}\underline{D}^T\underline{C}^T \tag{E-6}$$

Since

$$\frac{\partial}{\partial \underline{F}}\left[\underline{A} + \underline{B}\underline{F}^T\underline{C}\right]^k\underline{D} =$$

$$\sum_{l=1}^{n}\left[\underline{A} + \underline{B}\underline{F}^T\underline{C}\right]^{l-1}\frac{\partial}{\partial \underline{F}}\left[\underline{A} + \underline{B}\underline{F}^T\underline{C}\right]\left[\underline{A} + \underline{B}\underline{F}^T\underline{C}\right]^{k-l}\underline{D} \tag{E-7}$$

and from Eq (E-6)

$$\frac{\partial}{\partial \underline{F}}\left[\underline{A} + \underline{B}\underline{F}^T\underline{C}\right]\left[\left[\underline{A} + \underline{B}\underline{F}^T\underline{C}\right]^{k-l}\underline{D}\right] = \underline{B}\underline{D}^T\left[\underline{A} + \underline{B}\underline{F}^T\underline{C}\right]^{k-l^T}\underline{C}^T \tag{E-8}$$

so

$$\frac{\partial}{\partial \underline{F}}\left[\underline{A} + \underline{B}\underline{F}^T\underline{C}\right]^k\underline{D} =$$

$$\sum_{l=1}^{n}\left[\underline{A} + \underline{B}\underline{F}^T\underline{C}\right]^{l-1}\underline{B}\underline{D}^T\left[\underline{A} + \underline{B}\underline{F}^T\underline{C}\right]^{k-l^T}\underline{C}^T \tag{E-9}$$

QED

Appendix F

Derivation of Relationship Between $\underline{F}_y$ and $\underline{\lambda}_F$

The work presented in this appendix was developed by B. Porter (Ref 25) and presented by Prof. J. D'Azzo at the Air Force Institute of Technology, Wright-Patterson AFB, Ohio in 1978.

Generalized Control Canonical Form

The basic structure of the vector state equation representation of a system can be incorporated in the generalized control canonical form. This canonical form exhibits inherent properties of the system which are extremely useful in state feedback system design. This form is readily amenable to closed-loop system design which achieves a desired pole placement. While it is not the only form which can be used for pole placement, it has advantages for high order systems. The original state equation for the open-loop system is

$$\underline{x}(kT+T) = \underline{F}\underline{x}(kT) + \underline{G}\underline{u}(kT) \qquad \text{(F-1)}$$

where $\underline{x}(kT)$ is nx1, $\underline{F}$ is nxn, $\underline{G}$ is nxm, $\underline{u}(kT)$ is mx1, and the rank of $\underline{G}$ is m. By means of similarity transformations of $\underline{x}(kT)$ and $\underline{u}(kT)$ and by the addition of state feedback, this equation can be converted to the generalized canonical form, i.e., with all eigenvalues located at the origin, of the form

$$\underline{z}(kT+T) = \underline{F}_c\underline{z}(kT) + \underline{G}_c\underline{w}_c(kT) \tag{F-2}$$

where

$$\underline{F}_c = \text{block diagonal } (\underline{F}_{k_1}, \underline{F}_{k_2}, \cdots, \underline{F}_{k_m}) \tag{F-3}$$

and

$$\underline{G}_c = \text{block diagonal } (\underline{g}_{k_1}, \underline{g}_{k_2}, \cdots, \underline{g}_{k_m}) \tag{F-4}$$

The k_i in equations (F-3) and (F-4) are an ordered set of integers whose sum is n and are known as the control invariants. These control invariants are uniquely associated with the matrix pair $(\underline{F},\underline{G})$ and are identical to Kronecker's minimal column indices (Ref 25) for the singular pencil of matrices $(s\underline{I} - \underline{F},\underline{G})$ when they are ordered such that

$$k_1 \geq k_2 \geq \cdots \geq k_m > 0 \tag{F-5}$$

and

$$k_1 + k_2 + \cdots + k_m = n \tag{F-6}$$

The matrices $\underline{F}_{k_i}$ and $\underline{g}_{k_i}$ are of size $k_i x k_i$ and $k_i x1$ respectively, and they have the forms

$$\underline{F}_{k_i} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ & \vdots & & & & \\ 0 & 0 & 0 & & 0 & 1 \\ 0 & 0 & 0 & & 0 & 0 \end{bmatrix} \qquad \underline{g}_{k_i} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{F-7}$$

As shown, the submatrices $\underline{F}_{k_i}$ are companion matrices con-

taining ones on the superdiagonal and the remaining elements are zero, and the vectors $\underline{g}_{k_i}$ are null except for a one in its k_i row (which is its last row). An example of the canonical matrices for a system which has $k_1 = 3$ and $k_2 = 2$ is

$$\underline{F}_c = \left[\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad \underline{G}_c = \left[\begin{array}{c|c} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ \hline 0 & 0 \\ 0 & 1 \end{array}\right] \tag{F-8}$$

An important property of $\underline{F}_c$ is that it has an index of nilpotency (Ref 25) $\sigma = 3$, which corresponds to the largest value of k_i, i.e., $\sigma = k_1 = 3$.

The three transformations which may be applied to the original state equation (F-1) in order to achieve the generalized canonical form(F-2) are:

(1) A change of basis in the state variable $\underline{x}(kT)$, that is, $\underline{x}(kT) = \underline{T}\underline{z}(kT)$, where det $\underline{T} = 0$. Eq (F-1) then assumes the form

$$\underline{z}(kT+T) = \underline{T}^{-1}\underline{F}\underline{T}\underline{z}(kT) + \underline{T}^{-1}\underline{G}\underline{u}(kT) \tag{F-9}$$

(2) A change of basis in the control variable $\underline{u}(kT)$, that is, $\underline{u}(kT) = \underline{Z}\underline{v}_c(kT)$, where $\underline{Z}$ is mxm and Det $\underline{Z} = 0$. Applying this transformation changes equation (F-9) to the form

$$\underline{z}(kT+T) = \underline{T}^{-1}\underline{FTz}(kT) + \underline{T}^{-1}\underline{GZv}_c(kT)$$
$$= \underline{T}^{-1}\underline{FTz}(kT) + \underline{G}_c\underline{v}_c(kT) \qquad \text{(F-10)}$$

(3) The introduction of feedback

$$\underline{v}_c(kT) = \underline{w}_c(kT) - \underline{Lz}(kT) \qquad \text{(F-11)}$$

where $\underline{L}$ is an mxn matrix and $\underline{w}_c(kT)$ is an mx1 input vector. The matrix $\underline{L}$ is required to convert the plant matrix into the desired form. Substituting $\underline{v}_c(kT)$ from Eq (F-11) into Eq (F-10) yields the form

$$\underline{z}(kT+T) = \underline{T}^{-1}\underline{FTz}(kT) + \underline{T}^{-1}\underline{GZw}_c(kT) - \underline{T}^{-1}\underline{GZLz}(kT)$$
$$= (\underline{T}^{-1}\underline{FT} - \underline{T}^{-1}\underline{GZL})\underline{z}(kT) + \underline{T}^{-1}\underline{GZw}_c(kT)$$
$$= \underline{F}_c\underline{z}(kT) + \underline{G}_c\underline{w}_c(kT) \qquad \text{(F-12)}$$

It may not be necessary to apply all three transformations in every case; only the necessary transformations would be used to achieve the canonical form of the state equation. Also, there is no specified order in applying the transformations. It should be noted that transformations 1 and 2 produce equivalent matrices, that is, the control invariants do not change (Ref 25).

Control Invariants

The control invariants can be calculated from the nx(nm) controllability matrix

$$\underline{Q} = \left[\underline{G}, \underline{FG}, \underline{F}^2\underline{G}, \cdots, \underline{F}^{n-1}\underline{G}\right] \qquad \text{(F-13)}$$

It has been shown by Kalman (Ref 14) that when the rank of $\underline{G}$ is m, representing a multi-input system, that controllability can be evaluated from the nx(n-m+1)m matrix

$$\bar{\underline{Q}} = \left[\underline{G},\ \underline{FG},\ \underline{F}^2\underline{G}, \cdots, \underline{F}^{n-m}\underline{G}\right] \tag{F-14}$$

The columns of this matrix are

$$\bar{\underline{Q}} = \left[\underline{g}_1,\ \underline{g}_2, \cdots, \underline{g}_m,\ \underline{Fg}_1,\ \underline{Fg}_2, \cdots,\ \underline{Fg}_m,\ \underline{F}^2\underline{g}_1, \cdots,\ \underline{F}^2\underline{g}_m,\ \cdots,\ \underline{F}^{n-m}\underline{g}_m\right] \tag{F-15}$$

For a controllable system the rank $\underline{Q}$ = rank $\bar{\underline{Q}}$ = n. This means that the matrices $\underline{Q}$ and $\bar{\underline{Q}}$ have n linearly independent column vectors. Define the column vector $\underline{F}^i\underline{g}_j$ to be regular if it is linearly independent of the columns to its left in $\bar{\underline{Q}}$. A regular basis is a basis of the first n regular vectors contained in $\bar{\underline{Q}}$. These are rearranged to form

$$\hat{\underline{Q}} = \left[\underline{g}_1,\ \underline{Fg}_1, \cdots,\ \underline{F}^{k_1-1}\underline{g}_1,\ \underline{g}_2, \cdots,\ \underline{F}^{k_2-1}\underline{g}_2, \cdots,\ \underline{g}_m,\ \cdots,\ \underline{F}^{k_m-1}\underline{g}_m\right] \tag{F-16}$$

The k_i's are the control invariants of the system. If necessary, the columns of $\underline{G}$ are permuted so that the control invariants are in decreasing order, as given by Eq (F-3).

Transformation to Canonical Form - An Example

The transformation of the pair ($\underline{F}$,$\underline{G}$) to the canonical pair ($\underline{F}_c$,$\underline{G}_c$) can be obtained from a digital computer using

a coding based on the methods such as those of Aplevich (Ref 2). In order to illustrate the transformation using a third order plant, the method of Luenberger (Ref 18) based on his second canonical form is illustrated below for a system with two inputs:

$$\underline{x}(kT+T) = \begin{bmatrix} 3 & 5 & 1 \\ 0 & 0 & 1 \\ 0 & -2 & 3 \end{bmatrix} \underline{x}(kT) + \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 3 \end{bmatrix} \underline{u}(kT)$$

$$= \underline{Fx}(kT) + \underline{Gu}(kT) \tag{F-17}$$

$$\begin{vmatrix} \lambda - 3 & -5 & -1 \\ 0 & \lambda & -1 \\ 0 & 2 & \lambda - 3 \end{vmatrix} = (\lambda - 1)(\lambda - 2)(\lambda - 3) \tag{F-18}$$

Therefore

$$\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = 3$$

In reality this would be an unstable system, but as an example here it makes no difference.

$$\text{rank } \underline{\bar{Q}} = \text{rank } \left[\underline{G}, \ \underline{FG}\right]$$

$$= \text{rank} \left[\begin{array}{cc|cc} 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & 3 \\ 1 & 3 & 3 & 9 \end{array}\right] = 3 \tag{F-19}$$

Thus the system is controllable and the controllability index (Ref 25) $\upsilon = 2$. The first three columns of $\underline{\bar{Q}}$ are linearly independent, so $\underline{\hat{Q}}$ is formed as

$$\hat{\underline{Q}} = \left[\underline{g}_1,\ \underline{F}\underline{g}_1,\ \vdots\ \underline{g}_2\right] = \left[\begin{array}{cc|c} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 3 & 3 \end{array}\right] \tag{F-20}$$

Forming the inverse of this matrix yields

$$\hat{\underline{Q}}^{-1} = \begin{bmatrix} -3 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} \underline{e}_{11} \\ \underline{e}_{12} \\ \underline{e}_{21} \end{bmatrix} \tag{F-21}$$

The transformation matrix $\underline{T}^{-1}$ is formed by using the bottom row of each partitioned submatrix of $\hat{\underline{Q}}^{-1}$, as

$$\underline{T}^{-1} = \left[\begin{array}{c} \underline{e}_{12} \\ \underline{e}_{12}\underline{F} \\ \hline \underline{e}_{21} \end{array}\right] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \tag{F-22}$$

Now, using the transformation

$$\underline{x}(kT) = \underline{Tz}(kT) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \underline{z}(kT) \tag{F-23}$$

The plant equation becomes:

$$\underline{z}(kT+T) = \underline{T}^{-1}\underline{FTz}(kT) + \underline{T}^{-1}\underline{Gu}(kT)$$

$$= \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -2 & 3 & 0 \\ \hline 8 & 0 & 3 \end{array}\right] \underline{z}(kT) + \left[\begin{array}{c|c} 0 & 0 \\ 1 & 3 \\ \hline 0 & 1 \end{array}\right] \underline{u}(kT)$$

$$= \underline{\bar{F}}_c \underline{z}(kT) + \underline{\bar{G}}_c \underline{u}(kT) \tag{F-24}$$

In order to complete the transformation to the canonical form, it is necessary to use the additional transformations $\underline{u}(kT) = \underline{Z}\underline{v}_c(kT)$ and $\underline{v}_c(kT) = \underline{w}_c(kT) - \underline{L}\underline{z}(kT)$ so that the state equation becomes:

$$\underline{z}(kT+T) = (\underline{\bar{F}}_c - \underline{\bar{G}}_c \underline{Z}\underline{L})\underline{z}(kT) + (\underline{\bar{G}}_c \underline{Z})\underline{w}_c(kT)$$

$$= \underline{F}_c \underline{z}(kT) + \underline{G}_c \underline{w}_c(kT) \tag{F-25}$$

The $\underline{L}$ matrix is of order 2x3 and the $\underline{Z}$ matrix is 2x2, and hence the forms

$$\underline{L} = \begin{bmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \end{bmatrix} \qquad \underline{Z} = \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{bmatrix} \tag{F-26}$$

Inserting $\underline{Z}$ into Eq (F-24) yields

$$\underline{z}(kT+T) = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -2 & 3 & 0 \\ \hline 8 & 0 & 3 \end{array}\right] \underline{z}(kT)$$

$$+ \begin{bmatrix} 0 & 0 \\ \omega_{11} + 3\omega_{21} & \omega_{12} + 3\omega_{22} \\ \omega_{21} & \omega_{22} \end{bmatrix} \underline{v}_c(kT) \tag{F-27}$$

Since $\underline{\bar{G}}_c \underline{Z}$ should equal the generalized canonical form $\underline{G}_c$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \omega_{11} + 3\omega_{21} & \omega_{12} + 3\omega_{22} \\ \omega_{21} & \omega_{22} \end{bmatrix} \tag{F-28}$$

Equating coefficients yields

$$\underline{Z} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \tag{F-29}$$

The matrix $\underline{Z}$ will always be in the upper triangle form. Inserting $\underline{L}$ into

$$\underline{z}(kT+T) = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 3 & 0 \\ 8 & 0 & 3 \end{bmatrix} \underline{z}(kT) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \underline{w}_c(kT) - \underline{Lz}(kT) \tag{F-30}$$

yields

$$\underline{z}(kT+T) = \left[\begin{bmatrix} 0 & 1 & 0 \\ -2 & 3 & 0 \\ 8 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \end{bmatrix}\right] \underline{z}(kT) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \underline{w}_c(kT)$$

$$= \underline{F}_c \underline{z}(kT) + \underline{G}_c \underline{w}_c(kT) \tag{F-31}$$

Since $\underline{F}_c$ is in the generalized canonical form

$$\begin{bmatrix} 0 & 0 & 0 \\ l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 3 & 0 \\ 8 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{F-32}$$

Solving for $\underline{L}$ yields

$$\underline{L} = \begin{bmatrix} -2 & 3 & 0 \\ 8 & 0 & 3 \end{bmatrix} \tag{F-33}$$

The three transformations used are

$$\begin{aligned} \underline{x}(kT) &= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \underline{z}(kT) \\ &= \underline{Tz}(kT) \end{aligned} \tag{F-34}$$

$$\begin{aligned} \underline{u}(kT) &= \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \underline{v}_c(kT) \\ &= \underline{Zv}_c(kT) \end{aligned} \tag{F-35}$$

$$\begin{aligned} \underline{v}_c(kT) &= \underline{w}_c(kT) - \begin{bmatrix} -2 & 3 & 0 \\ 8 & 0 & 3 \end{bmatrix} \underline{z}(kT) \\ &= \underline{w}_c(kT) - \underline{Lz}(kT) \end{aligned} \tag{F-36}$$

which transformed the original system representation to the generalized control canonical form.

Pole Placement by State Feedback

Given an open-loop plant represented by

$$\underline{x}(kT+T) = \underline{F}\underline{x}(kT) + \underline{G}\underline{u}(kT) \qquad \text{(F-1)}$$

the intent is to determine a state feedback control law

$$\underline{u}(kT) = \underline{K}\underline{x}(kT) \qquad \text{(F-37)}$$

which will produce specified eigenvalues for the feedback system. The design procedure is not to solve for $\underline{K}$ directly but to make use of the generalized canonical state equation (F-11). A feedback represented by

$$\underline{w}_c(kT) = \underline{H}\underline{z}(kT) \qquad \text{(F-38)}$$

produces

$$\begin{aligned} \underline{z}(kT+T) &= \underline{F}_c\underline{z}(kT) + \underline{G}_c\underline{H}\underline{z}(kT) \\ &= (\underline{F}_c + \underline{G}_c\underline{H})\underline{z}(kT) \\ &= \underline{F}_d\underline{z}(kT) \end{aligned} \qquad \text{(F-39)}$$

where $\underline{F}_d$ is the desired feedback system matrix. It is selected to have the form

$$\underline{F}_d = \text{block diagonal } (\underline{c}_{k_1}, \underline{c}_{k_2}, \cdots, \underline{c}_{k_m}) \qquad \text{(F-40)}$$

where each $\underline{c}_{k_i}$ is a matrix block which has the same size as the blocks in $\underline{F}_c$ and is in companion form. In Eq (F-39) the matrices $\underline{F}_c$, $\underline{G}_c$, and $\underline{F}_d$ are now known and $\underline{H}$ can be evaluated by equating the elements of $(\underline{F}_c + \underline{G}_c\underline{H})$ and $\underline{F}_d$. A formal expression for $\underline{H}$ can be obtained since

$$\underline{G}_c^T\underline{G}_c = \underline{I}_m \qquad \text{(F-41)}$$

This leads to

$$\underline{H} = \underline{G}_c{}^T(\underline{F}_d - \underline{F}_c) \tag{F-42}$$

In order to return to the original state space, it is observed that

$$\begin{aligned} \underline{u}(kT) &= \underline{Zv}_c(kT) \\ &= \underline{Zw}_c(kT) - \underline{ZLz}(kT) \\ &= \underline{Z}(\underline{H} - \underline{L})\underline{z}(kT) \\ &= \underline{Z}(\underline{H} - \underline{L})\underline{T}^{-1}\underline{x}(kT) \\ &= \underline{Kx}(kT) \end{aligned} \tag{F-43}$$

Thus, the state feedback matrix is

$$\underline{K} = \underline{Z}(\underline{H} - \underline{L})\underline{T}^{-1} \tag{F-44}$$

The advantages of using the generalized canonical form are illustrated by continuing the example.

Suppose the desired feedback system eigenvalues are $\lambda_1 = 1$, $\lambda_{2,3} = .5 \pm j.5$ in order to make the system stable. The factors of the characteristic polynomial of order 2 and 1 respectively are

$$(\lambda - .5 + j.5)(\lambda - .5 - j.5) = \lambda^2 - \lambda + .5 \tag{F-45}$$

and

$$\lambda - 1 \tag{F-46}$$

The corresponding matrix blocks of the desired feedback sys-

tem matrix $\underline{F}_d$ are the companion form matrices associated with these characteristic polynomials

$$\underline{c}_{k_1} = \begin{bmatrix} 0 & 1 \\ -.5 & 1 \end{bmatrix} \tag{F-47}$$

$$\underline{c}_{k_2} = \begin{bmatrix} 1 \end{bmatrix} \tag{F-48}$$

From Eq (F-42)

$$\underline{H} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{bmatrix} 0 & 1 & 0 \\ -.5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right]$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ -.5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -.5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{F-49}$$

It is seen that the rows of $\underline{H}$ are made up of the coefficients of the characteristic polynomials and zero terms. The feedback matrix $\underline{K}$ is then determined from Eq (F-44)

$$\underline{K} = \underline{Z}(\underline{H} - \underline{L})\underline{T}^{-1}$$

$$= \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \left[\begin{bmatrix} -.5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -2 & 3 & 0 \\ 8 & 0 & 3 \end{bmatrix} \right] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1.5 & -2 & 0 \\ -8 & 0 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 19.5 & -2 \\ -2 & -6 & 0 \end{bmatrix} \tag{F-50}$$

Inserting $\underline{u}(kT) = \underline{\underline{K}}\underline{x}(kT)$ into Eq (F-17) yields

$$\underline{x}(kT+T) = \underline{F}\underline{x}(kT) + \underline{\underline{G}}\underline{\underline{K}}\underline{x}(kT)$$

$$= (\underline{F} + \underline{\underline{GK}})\underline{x}(kT) \tag{F-51}$$

Substituting the calculated matrices into Eq (F-51) yields

$$\underline{x}(kT+T) = \left[\begin{bmatrix} 3 & 5 & 1 \\ 0 & 0 & 1 \\ 0 & -2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 6 & 19.5 & -2 \\ -2 & -6 & 0 \end{bmatrix} \right]$$

$$= \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & -.5 & 1 \end{bmatrix} \underline{x}(kT) \tag{F-52}$$

To check this

$$\begin{vmatrix} \lambda - 1 & 1 & -1 \\ 0 & \lambda & -1 \\ 0 & .5 & \lambda - 1 \end{vmatrix} = \lambda(\lambda - 1)^2 + .5(\lambda - 1)$$

$$= \lambda^3 - 2\lambda^2 + \lambda + .5\lambda - .5$$

$$= \lambda^3 - 2\lambda^2 + 1.5\lambda - .5$$

$$= (\lambda - 1)(\lambda - .5 + j.5)(\lambda - .5 - j.5)$$

Thus, the eigenvalues are 1, $.5 \pm j.5$ as desired. This concludes the work presented on the generalized canonical control form.